In this section we prove theorem 3.18 for the special case where \( \mathcal{F} \) is a sequence such that each \( \mathcal{F}_k \) is countably complete: if \( \mathcal{F} \) is a sequence such that each \( \mathcal{F}_k \) is a countably complete \( \text{IFP}(\alpha, \beta) \)-ultralimit then \( \mathcal{F} \) is an ultralimit sequence in \( \text{IFP} \). To do so we need to show that for all \( \alpha, \beta \in \mathcal{F} \), \( \mathcal{F} \) is a sequence in every \( \text{IFP} \)-ultralimit. Then we have reduced the problem from our claim \( \text{IFP} \) to one about the countable\-complete hierarchy: we can use recursion over \( \alpha \) to analyze the structures \( \mathcal{F}_\alpha \) for all \( \alpha \leq \beta \). We can assume that \( \alpha \) is countably complete above \( \alpha \) by taking transitive reductions if necessary, that the following lemma will accomplish our purpose.

**Lemma:** Suppose \( \alpha \) is an ordinal and \( \mathcal{F} \) is a sequence such that each \( \mathcal{F}_k \) is a countably complete \( \text{IFP}(\alpha, \beta) \)-ultralimit or else is a countable \( \text{IFP} \)-ultralimit. Then

1. \( \mathcal{F} \) is an ultralimit sequence in \( \prod_{\alpha < \beta} \mathcal{F}_\alpha \).
Lemma 4.1.2 is a weak form of the statement that \( S_{31} \) can be uniformized. The full statement (Lemma 6.4(c)) asserts that \( S_{31} \) codes for \( S' \) except for obvious reasons we delay it until we have been able to define the \( S_{31} \) code of \( \lambda_0 \). Lemma 4.1.2 is in a consequence of the existence of \( S_{31} \) code and will be needed as an induction hypothesis before we define the code.

Lemma 4.1 is proved by induction on \( \alpha \) and \( \beta \); we assume that \( 0 < \alpha \) (and, in fact, \( \alpha \), in turn, is true for \( \alpha' < \alpha \) whenever \( \alpha' < \alpha \) is a proper initial segment of \( \alpha \)) or \( \alpha = \beta \) and \( \alpha' < \beta \); and we will prove that 4.1.2 (Lemma 4.1.1) also holds for \( \alpha' \). Note that the induction hypothesis implies that \( S_{31} \) codes \( \beta \) and that \( \beta \) is an ordinal sequence in \( S_{31} \).

Cubes were first introduced by Lemma 4.1.3 for the \( \beta \)-hierarchy of the constructive sets in order to analyze the definable subsets of \( \alpha_0 \). To define the \( S_{31} \) code of \( \alpha_0 \), we took the least ordinal \( \alpha_1 \) such that there is a subset \( \alpha \) of \( S_1 \) which is \( S_1 \)-definable over \( \alpha_1 \), but not a member of \( S_1 \). So then defined the \( S_{31} \) code, \( \alpha_1 \subseteq S_{31} \), of \( \alpha_1 \) to be essentially the \( S_1 \) theory of \( S_{31} \) with constants from \( \alpha_1 \). It turned out that \( \alpha_1 \) codes the entire \( S_1 \) theory of \( S_{31} \) on a \( S_{31} \) formula can be regarded as an \( S_{31} \) formula involving the universal \( S_{31} \) predicate \( A_1 \), and hence as an \( S_{31} \) formula over the structure \( (S_{31}, A_1) \). The \( S_{31} \) code of \( \alpha_1 \) is then obtained by taking the \( S_{31} \) code of \( (S_{31}, A_1) \). So removing we get all formulas over \( \alpha_1 \), even though the only formulas we work with directly are only \( \alpha_1 \) over
the appropriate structures.

The $\Sigma^1_1$ case for $\Delta^1_1$ are constructed analogously, but because we have
the extra predicate $P$, the notion of definability is more complicated. This
appears at two places. First, if $m = \langle n, A \rangle$ for some pair $\langle n, A \rangle$ then
$P(x, m)$ may contain new sets not in $\Sigma^1_1$ of the form $x = \langle k, \phi(x) \rangle$ where $x$


Instead of going straight to $\Sigma^1_1$ sets we will define a $\Sigma^1_2$ code
$Q_n$ so as to encode such sets. The first part of this section defines the $\Sigma^1_2$
sets and proves that it exists.

The second change comes in the notion of definability, which must now
include definability from the predicate $P$. We first define "usability
structures" as enriched analogs of Tarski's usable structures, and then
define $\Sigma^1_2$ formulas by allowing $P$ as an additional generalized quantifier.

The $\Sigma^1_2$ case of a "usable structure" $\mathcal{A}$ is then defined as to be a
"usable structure extending the $\Sigma^1_2$ theory of $\mathcal{A}$. Since the $\Sigma^1_2$


"usable" we can then proceed as with the constructible sets.

Since $\Sigma^1_1$ case $Q_n$ of $\mathcal{A}$ was the $\Sigma^1_2$ code of the $\Sigma^1_1$


An induction over the (newly defined) $\Sigma^1_2$ construction.

Notice that if we never define a $\Sigma^1_2$ formula for $\phi \iff \phi^\mathcal{A}$


$\phi^\mathcal{A}$ formula can be constructed as a $\Sigma^1_2$ formula over the $\Sigma^1_2$


$\phi^\mathcal{A}$ code. Since we have assumed that Lemma 5.1 holds for any proper


initial segment of $\mathcal{A}$, we can assume that $M(n \in \mathcal{A}$ and that $\langle n, A \rangle \leq A


for all $\langle n, A \rangle \in \text{dom}(\mathcal{A})$. If $\langle n, A \rangle \leq A$ for all $\langle n, A \rangle$ then $\phi^\mathcal{A}$ is


"usable". In the trivial case the $\Sigma^1_2$ code $Q_n$ of $\mathcal{A}$ is defined to be


$\phi^\mathcal{A}$. Throughout the rest of this section we assume that $\text{dom}(\mathcal{A})$
For a largest member \( a \), and that \( 4n \cdot a = a \). Let \( F = P(a) \).

**Definition:** A set \( X \) is in \( P(a) \) if there is a \( b \) and \( F = P(b) \) such that \( X \subseteq (b \in \text{dom}(F); F(x) \in \mathcal{P}^F) \). A function is said to be \( P(a) \) if its graph is.

The class \( P(a) \) is clearly closed under Boolean operations. Since \( P(a) \subseteq \text{dom}(P(a)) \), \( F \) is normal on all functions definable in \( P(a) \). (It follows that \( \text{dom} P(a) \) is closed under quantification bounded by \( a \).) I.e., if \( X \subseteq P(a) \), then so is \( Y = \{y \in a \mid \exists z F_z(y) \} \). For see this suppose \( X \subseteq P(a) \). Set \( \{z \mid \exists \langle x, z \rangle_F \in a \} \neq P(a) \). Set \( W \cdot x = \{x \in P(a) \mid \exists \langle x, y \rangle_F \in a \} \).

If \( F \cdot x = \{y \in P(a) \mid \exists \langle x, y \rangle_F \in a \} \), then \( W \cdot x = \{x \in P(a) \mid \exists \langle x, y \rangle_F \in a \} \).

\[
\begin{align*}
|z| & \leq 2, \exists x \in a \text{ if } \langle x, z \rangle \in a. \\text{ If } \langle x, z \rangle \text{ and } \langle x', z \rangle \text{ both hold if } \langle x, z \rangle \text{ then } \langle x', z \rangle \text{ and } \langle x, z \rangle \text{ both hold.}
\end{align*}
\]

If there is no \( \langle x, y \rangle \), then we again trivially define the \( \text{dom} P(a) \) to be \( \langle x, y \rangle \). Suppose that there is a \( \langle x, y \rangle \). Then there is a \( \langle x, y \rangle \) of the form \( \langle x, y \rangle \in a \). Since \( \langle x, y \rangle \) is well ordered we can assume that \( x \leq \alpha \). Let \( a^0 \) be the least such ordinal.

Then \( a^0 = \alpha \). Since \( \rho < x \) and \( F \cdot 0 = \alpha \), then by normality there is \( \rho = \alpha \) such that \( \langle \alpha, \rho \rangle \in a \). Since \( \langle \alpha, \rho \rangle \in a \), then \( \rho = \alpha \). By Lemma 1.1.11, \( \rho \) is equal to \( \alpha \) is \( \langle x, y \rangle \). So there are only two possible cases: \( p_x = \alpha \) and \( \alpha = \gamma \), or defined in \( \mathcal{P}^a \). In the case \( p_x = \gamma \), we can set \( a^0 = \gamma \). In the case \( p_x = \alpha \), which we will consider first, we will actually show that every
Proposition: If $p_x = x$, then there is a map $\eta : \mathbb{R}^+ \to \mathbb{R}$ taking $x$ onto $\mathbb{R}$.

Proof: Let $\mathbb{R} = (\mathbb{R}^+ \cup \{0\})$ be the canonical ordering. If $x$ is a non-archimedean integer, say $x = k \cdot \omega$, for some $k \in \mathbb{R}^+$, then $x = k \cdot \omega$ is a limit element. Now, $\omega(x) = x \cdot \omega(x)$, and since $x < \omega(x)$, $\omega(x) > x > x \cdot \omega(x)$. By the induction hypothesis, there is a limit element $y \in \mathbb{R}$ for every $y' \in \mathbb{R}$ and in particular $\omega(x) = x \in \mathbb{R}$. Let $b$ be any $\omega$-map of a copy $\mathbb{R}$ in $\mathbb{R}$.

We claim that $x < b$. Let $x' \in \mathbb{R}$ be such that $x' = b \cdot \omega$. Then for $x = x' \cdot \omega$, we have $\omega(x') = x' \cdot \omega(x')$, so $\omega(x) = x' \cdot \omega(x') = x' \cdot x = x$, which implies $x < b$. Thus, for $x = x' \cdot \omega$, $x < b$. (If there are $x' \cdot \omega$ and $b' \cdot \omega$, such that $x$ is a well-ordering of $b$ of order type $\omega$), then $x'$ is a well-ordering of $x'$ of order type $\omega$. Then $\omega(x') = x' \cdot \omega(x')$, and $\omega(x') = x'$ is in $\omega(x')$, and, since by proposition 1.3 $x \leq x'$, $x' = x$ in $\mathbb{R}$. \qed
We will see proposition 1.3 to show that every subset of \( N \) in \( \mathcal{L}_{N} \) is the class \( \gamma \). \( \mathcal{L}_{N} \) is a \( \Sigma_{2}^{1} \) subset of \( N \). Proposition 1.4 then follows immediately.

We need to see first that every ordinary subset \( N \) of \( \mathcal{L}_{N} \) is in \( N \). For if \( N \) is in \( N \), then \( N \) is in \( \mathcal{L}_{N} \). Now let \( \gamma \) be \( \gamma \) and \( \alpha \) be \( \alpha \). Then \( \alpha \) is in \( \mathcal{L}_{N} \). The class \( \gamma \) is definitely closed under boolean combinations, so we only need to verify 1.3(1) and 1.3(2). Proposition 1.3(1) asserts that \( \gamma \) is closed under existential quantification. \( \gamma = \forall \alpha \, \exists \beta \, \gamma \), then so is \( \forall \alpha \, \exists \beta \, \gamma \).

\[
\gamma \vdash \exists \alpha \, \exists \beta \, \gamma
\]

and so \( \exists \alpha \, \exists \beta \, \gamma \) is in \( \mathcal{L}_{N} \). Class \( \mathcal{L}_{N} \) is closed under quantification over \( \gamma \).

Proposition 1.3(2) asserts that \( \gamma \) is closed under \( \mathcal{L} \). That is, if \( \mathcal{L} \) is in \( \mathcal{L} \) then \( \mathcal{L} \) is in \( \mathcal{L} \).

\[
\mathcal{L} \vdash \exists \alpha \, \exists \beta \, \mathcal{L}
\]

and since it is easy to see that \( \exists \alpha \, \exists \beta \, \mathcal{L} \) is in \( \mathcal{L} \), the set \( \mathcal{L} \) is in \( \mathcal{L} \). This set is the image under \( \mathcal{L} \) of

\[
\exists \alpha \, \exists \beta \, \gamma
\]

and since it is easy to see that \( \exists \alpha \, \exists \beta \, \gamma \) is in \( \mathcal{L} \), the set \( \gamma \) is in \( \mathcal{L} \). For a \( \mathcal{L} \) set \( \mathcal{L} \). But \( \gamma \) is in \( \mathcal{L} \).

\[
\mathcal{L} \vdash \forall \alpha \, \forall \beta \, \gamma
\]
and to use that (*), it is enough to show that the functions

\[ h(x) = \langle x, (\lambda y . \alpha(y)(x)) \rangle \in \mathfrak{A}_\mathbb{Z} \] 

at least modulu \( \mathfrak{A}_\mathbb{Z} \) for all \( x \) in \( \mathbb{Z} \). Thus \( h(x) = \langle x, (\lambda y . \alpha(y)(x)) \rangle \in \mathfrak{A}_\mathbb{Z} \)

Then \( \varpi \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \)

since \( \mathfrak{A}_\mathbb{Z} \) is a \( \mathfrak{A}_\mathbb{Z} \) ultrafilter.

\[ \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \] 

and hence \( \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \) and \( \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \) by Proposition 4.3.

Thus Lemma 4.1 is true for \( \mathfrak{A}_\mathbb{Z} \).

This completes the case \( p_0 = -\). Now we consider the case that \( p_0 = -\) in \( \mathfrak{A}_\mathbb{Z} \) the term "\( -\)" all mean \( -\) as defined in \( \mathfrak{A}_\mathbb{Z} \).

4.3 Proposition: If \( p_0 = -\) then there is a map \( \tilde{ \varphi } : \mathfrak{A}_\mathbb{Z} \rightarrow \mathfrak{A}_\mathbb{Z} \) taking \( p_0 \) onto \( \mathfrak{A}_\mathbb{Z} \).

Proof: As in the proof of Proposition 4.2, let \( \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \). Then \( \alpha(x) = \beta(x) = \gamma(x) \) and \( \lambda x . \alpha(x) \) are \( \mathfrak{A}_\mathbb{Z} \) functions \( \gamma(x) \) and \( \lambda x . \alpha(x) \).

\[ \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \] 

let \( f \) be such that \( a = b(f(x) \in \mathfrak{A}_\mathbb{Z} \) for all \( a \leq x \) and let \( \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \) be as moreover in \( \mathfrak{A}_\mathbb{Z} \) of the functions \( \lambda x . \alpha(x) \) such that \( \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \).

This proposition extends since \( \lambda x . \alpha(x) = \mathfrak{A}_\mathbb{Z} \).

Before \( \lambda x . \alpha(x) \rightarrow \mathfrak{A}_\mathbb{Z} \), then \( \alpha(x) \in \mathfrak{A}_\mathbb{Z} \).

4.4 Definition: If \( p_0 = -\) then the \( \mathfrak{A}_\mathbb{Z} \) case \( \mathfrak{A}_\mathbb{Z} \) of \( \mathfrak{A}_\mathbb{Z} \) is

\[ \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \subseteq \mathfrak{A}_\mathbb{Z} \] 

where \( \lambda x . \alpha(x) \rightarrow \mathfrak{A}_\mathbb{Z} \) for \( \mathfrak{A}_\mathbb{Z} \).

It may be that \( \lambda \) is not a member of the range of \( \alpha \) as \( \lambda \) is defined above. In that case the definition of \( \alpha \) should be modified to the sentence...
The purpose of the function $f$ will be presented in connection with $X_n$ codes.

**Proposition:** $A_n \times X_n \rightarrow X_n$, $A_n$ is semantically, and there is function $k$ which is $A_n$ over $(P, A_n)$ such that for all $a \in A_n$ and $1 \leq k \leq n - 1$:

$k \in (a \in A_n) \Leftrightarrow A_n^a \rightarrow A_n^a$

**Proof:** The proof that $A_n$ is $X_n$ is exactly the same as the proof in Proposition 4.4 that the refractive sets are $X_n$.

The required function $k$ can be defined by $k(t, a) = q(a)$. Then:

$(x, t, a) \in A_n^a \Leftrightarrow q(a) = x(t, a)$

for all $x \leq a^*$ since $a \leq a^*$.

If $x \leq a$, then $k = A_n$ on $X_n \times [a, b]$ and hence $(x, t, a) \in A_n^a$.

Also, if $x \leq a$, then $(x, t, a) \in X_n$ because $k \in A_n^a$ and $q(a) = x(t, a) \leq a$.

If $x \leq a$, then $(x, t, a) \in A_n^a$ and hence $k \in A_n^a$.

Since $A_n$ is semantically:

If $x$ is in $X_n$ and $a$ then we can recover $a$ from $x$ by a straightforward process. If $x'$ is an arbitrary character then the same process is successful in obtaining $a' = a$ and $x''$ is in the $X_n$ code of $a''$. If it is successful then $x''$ is called the $X_n$ encoding of $x$. The next proposition gives conditions under which $x''$ can be decoded, it will be used later in a generalization to $X_n$ code. The proof is easy.
A.8 Proposition: Suppose \( \vec{F} = (a, b, v^2) \) is the \( \mathbb{C} \) code for \( a \).

1. If \( a \) is \( \mathbb{C} \), then the decaying \( a \) or \( b \) is well-defined. If \( b \) is well-defined then \( b \) is the \( \mathbb{C} \) code.
2. If \( a \to b \) and \( b \) is well-defined then \( a \) extends to \( a \to b \).

3. If \( a \to b \) and \( b \) is well-defined then \( a \) extends to \( a \to b \).
functions for measures on \( \mathcal{B} \). In our applications, \( (b, A) \) will be the \( \mathcal{B} \)-set of \( A \) and the regularizing structure will be the transitive closure of a fragment of \( \mathcal{B} \).

\[ A \to \mathcal{B} \]

\textbf{A.3 Proposition:} Suppose \( b \in \mathcal{B} \) and \( \mathcal{B} \) regularizes \( A \). Then

\[ A \subseteq \mathcal{B} \]

\[ \text{There is a } \mathcal{B} \text{-function } H : \mathbb{R}^2 \to \mathbb{R} \text{ linear in } b, \text{ i.e., such that } \]

\[ \mathbb{R}^2 \to \mathbb{R} \text{ is the identity.} \]

\[ \text{There is a well ordering } \leq \text{ of } \mathcal{B} \text{ which is } \mathcal{B} \text{-over } (b, A) ; \text{ furthermore if } \]

\[ a \in \mathcal{B}, \text{ then } \]

\[ a = \mathcal{B}(g) \text{ for some } g \in \mathcal{B} \text{ in } \mathcal{B} \text{ over } (b, A). \]

\[ \text{Proof:} \quad \text{Suppose } a \in \mathcal{B}(g) \text{ if } a \subseteq b. \text{ There are } \]

\[ a \subseteq b, \text{ such that } a \subseteq b \text{ and } \]

\[ b = \mathcal{B}(g) \text{ in } \mathcal{B} \text{ over } (b, A). \]

\[ \text{The rest of the proposition follows from the fact that } \]

\[ \mathcal{B}(g) = \mathcal{B}(f), \text{ where } \mathcal{B}(g) \text{ is the } \mathcal{B} \text{-regularizing function in } \mathcal{B} \text{ over } \mathcal{B}(f) \text{ in } \mathcal{B}. \]

\[ \mathcal{B} \]

\[ \text{A.3.1 Corollary: } \text{If } (b, A) \text{ is weakly measurable, then it has a universal } \mathcal{B} \text{-}

\[ \text{regularizing function.} \]

If we are given \( (b, A) \) and told the \( \mathcal{B} \)-regularizing set using truth is

\[ \mathcal{B} \]

then we can construct \( \mathcal{B} \) : it is isomorphic to the set of

\[ \text{expressions classes } [z], \text{ where } x \to \mathcal{B}(y) \text{ is } \mathcal{B} \text{ in } \mathcal{B} \text{ over } (b, A), \text{ with the}

\[ \text{membership relations } [x] \subseteq [y] \text{ if } \]

\[ \text{"ind \( b \) in } \mathcal{B}(y) \text{ for } b \in \mathcal{B}. \text{ If we are given}

\[ \text{the regularizing function required by } (b, A) \text{ then there is a } \mathcal{B} \text{ formula asserting that } \]

\[ \mathcal{B} \text{ can be constructed (but not that it is well-}

\[ \text{formed).} \]
Proposition: Suppose that \( \Sigma(\lambda) \Rightarrow (\Sigma',\lambda) \).

(i) If \( (\Sigma',\lambda) \) is weakly well-founded and \( \lambda \in \Sigma' \), \( \lambda \) is \( \Sigma' \)-elementary then \((\Sigma,\lambda)\) is weakly well-founded.

(ii) If \( \lambda \) is weakly well-founded then \( \lambda = (\Sigma',\lambda') \); provided either that \( \lambda \) is \( \Sigma' \)-elementary and the \( \delta' \) constructed from \((\Sigma',\lambda')\) is well-founded, or that \( \lambda \) is an invariant ultrapower by possibly complete ultradiscretions.

In either case, \( \lambda \) can be extended to a map \( \delta' : \delta' \rightarrow \delta' \).

The extension \( \delta' \) enables us, in particular, to extend a map \( \iota : (\Sigma,\lambda) \rightarrow (\delta',\lambda') \) to a map from \( \delta' \) to \( \delta' \). It is not clear that \( \delta'(\mu) \) is always equal to \( \mu' \) but it is equal in the most important case, when \( \delta'(\mu) = \mu' \) and \( \lambda \) is an invariant ultrapower by ultradiscretions from \( \eta \) on continua below \( \lambda \), or \( \mu' \) is ultrapower by measures below \( \lambda \); this is sometimes from the regularity of \( \lambda \). If \( \iota \) is a \( \Sigma' \)-ultrapower then any member of \( \delta' \) is of the form \( \delta'(g) \) where \( g \) is a \( \Sigma' \)-function in \( (\Sigma,\lambda) \). If \( \delta'(g) = \mu'(\lambda) \) then \( \delta'(g) < \lambda \) has measure 1, but it is easy to check that if \( \mu'(\lambda) \) is defined to be \( \delta'(g) \) whenever \( \delta'(g) \) is a \( \Sigma' \)-function in \( (\Sigma,\lambda) \). The exact \( \Sigma' \)-code of \( \delta' \) will be a "well-founded structure" \((\Sigma,\lambda)\).
that \( \mu \rightarrow \text{cod}(\mathbb{H}) \) is the \( \mathbb{H} \) projection of \( \mathbb{H} \) and \( \mathbb{K} \) is a subset of \( \mathbb{H} \) containing the \( \mathbb{K} \) vertices of \( \mathbb{H} \).

The full \( \mathbb{H} \) code will also encode a part of \( \mathbb{H} \). We will not need to consider \( (\mathbb{H}, \lambda) \) for \( \lambda \neq \mu \), since this is already encoded by \( \mathbb{K} \) and \( \mathbb{K} \) is isomorphic to \( \mathbb{H} \).

The inclusion of \( \mathbb{H} \) in \( \mathbb{K} \) is straightforward, except for the possibility that \( \lambda \geq \mu \) for some \( \lambda \) assigned to \( \mathbb{H} \) or that \( \lambda > \mu \), in which case we take \( \lambda = \mu \).

Most of the main structures in a "dual" structure (e.g., \( \mathbb{H} \)) is encoded to code \( \mathbb{H}(\mathbb{H}) \). The coding is accomplished by using the regularizing function \( \beta \) (free Definition 6.0.0); the requirement that \( \gamma > \alpha \) (\( \mathbb{H}(\mathbb{H}) \)) rather than \( \alpha \) (\( \mathbb{H}(\mathbb{H}) \)) ensures that the compatibility functions for \( \mathbb{H} \) are available.

6.15 Definition: A structure \( (\mathbb{H}, \mathbb{K}) \) is "dual" if
\[ \mathbb{H}(\mathbb{H}) \] is a substructure of \( \mathbb{H} \) with regularizing function \( \beta \). Let
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]
\[ \mathbb{H}(\mathbb{H}) = \mathbb{H} \]
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]

6.14 Proposition: \( \mathbb{H} \) cubes are "duals".

6.15 Definition: \( \mathbb{H}(\mathbb{H}) \) formula. Suppose \( \mathbb{H}(\mathbb{H}) \) is a substructure of \( \mathbb{H} \) with regularizing function \( \beta \). Let
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]
\[ \mathbb{H}(\mathbb{H}) \subseteq \mathbb{H} \]

6.16 Proposition: \( \mathbb{H} \) cubes are "duals".
are the formulas of the form

$$\Phi(x)$$

where $$\Phi$$ is a $$\Sigma_1$$ formula over $$\mathcal{B}$$.

If $$\Phi$$ is in the form (1) then we will write $$\Phi(x)$$ for the formula

$$\forall x \in X \cdot \exists y \in Y \cdot \Phi(x,y)$$

where there is a universal $$\Sigma_1$$ formula there is a universal $$\Sigma_1$$

formulas that is, a $$\Sigma_1$$ formula $$\Phi$$ such that for any $$\Sigma_1$$ formula $$\Phi$$ there

is an $$\alpha(x,y)$$ such that for all $$x \in X \cdot \Phi(x,y)$$ holds iff $$\Phi(x,y)$$ holds.

If $$\Phi(x,y)$$ holds then the $$\Sigma_1$$ formulas are interesting only if $$\Phi$$ is a $$\Sigma_1$$

intermediate sequence for $$\mathcal{B}$$, and we will assume that unless stated

otherwise.

A.1 Proposition: If $$\Phi(x)$$ holds then every $$\Sigma_1$$ function with domains in some

$$f \in F$$ is a member of $$\mathcal{B}$$.

Proof: Suppose $$h(x) \rightarrow y \cdot h(x)$$ is defined by $$h(x) \rightarrow \Phi(x,y)$$ where $$\Phi$$

is arbitrary. Then by the $$\Sigma_1$$ normality of $$\Phi(x,y)$$ the $$\Sigma_1$$ function

$$h(x) \rightarrow y$$ is defined by $$h(x) \rightarrow \Phi(x,y)$$ and $$h(x) \rightarrow y$$ and $$F \in F$$

must be bounded by some $$\omega_1 \cdot h(x) \rightarrow y$$ and $$F \in F$$

$$x \rightarrow y \exists \xi \in F$$ or $$x \rightarrow y$$.

If $$\Phi(x)$$ holds then the formula $$\Phi(x,y)$$ will depend on the

particular $$\Sigma_1$$ formula $$\Phi$$ and to define $$\Phi(x,y)$$ in equation (1) of 1.15.

The exact choice of $$\Phi$$ does not matter; we will assume that each formula $$\Phi$$

corresponding with some particular $$\Phi$$ so that $$\Phi(x)$$ has the property

specified in the following proposition.
8.5 Proposition: any $\mathbb{Z}_p$ formula $\varphi$ can be written in the form of equation (1) at $s$. It is such that for all $a, b, c$ and $\varphi$ if $\varphi(a)$ holds then $\varphi(b)$ holds for all $b$ with $a(b') = b$.

Proof: Suppose $\varphi$ holds. If $\exists x \varphi(x)$ holds then $\varphi(x)$ holds for all $x$ in $\mathbb{Z}_p$. Thus, $\varphi(x)$ holds for all $x$ in $\mathbb{Z}_p$.

8.6 Corollary: Suppose $\varphi$ and $\psi$ are $\mathbb{Z}_p$ formulas and $\varphi$ is a $\mathbb{Z}_p$ formula. Then $\exists y \psi(y) \supset \varphi$ and $\psi(0)$ are $\mathbb{Z}_p$ formulas.
A 10. Lemma \( \exists \) choice functions exist, i.e., if \( \Phi \) is a \( \mathfrak{U} \) formula then there is a \( \mathfrak{U} \) function \( a \) such that for all \( x \in A \), \( a(x) \) is defined whenever \( \Phi(x,a(x)) \) and \( \Phi(x,b(x)) \) holds whenever \( b(x) \) is defined.

Proof: Suppose \( \Phi(x,y) \) is defined by \( \exists b \left( \Phi(x,b,y) \land \Phi(x,a(y)) \right) \), where \( \Phi(x,y,z) \) iff \( \Phi(x,b,y,z) \). We set \( a(x) = y \) iff there is a \( z \) such that

1. \( a(x) \in A \), \( \Phi(x,a(x),a(x)) \),
2. \( a(x) \in A \), \( \Phi(x,a(x),a(x)) \),
3. \( a(x) \in A \), \( \Phi(x,a(x),a(x)) \).

This clause (1) asserts that \( a(x) \) is chosen as small as possible for any choice of \( y \) and \( z \), clause (2) asserts that the function \( \Phi \) is true of \( (x,a(x),a(x)) \) is as small as possible (modulo \( x \) and \( z \)) for this \( y \) and any choice of \( x \), and clause (3) asserts that \( y \) is as small as possible for this choice of \( x \) and in function \( x \). Clause (1) can be
and hence (in $Z^+$) Theorem (4) and (5) can also be put into $\tilde{\Psi}$ form and hence so are $\delta$, which is evidently the required function.

Since there is a universal $Z^+$ formula, Lemma 4.10 yields a universal $Z^+$ closure function. We will shortly use $\delta$ to denote this function.

In the case when $F(\gamma) = \delta$, so that the $Z^+$ formulas are just the $\Gamma_0$ formulas, we will write $\gamma_\alpha$ for $\gamma \in \Delta$ and $\delta$ in the $\Gamma_0$ formulas.

We can easily verify that 4.17, 4.18, and 4.19 also hold in this case.

2.9. Applications of Part (6) (statement).

We shall outline here the theory of iterated ultrameasures of a function structure $\mathcal{M} = (M, \mathcal{A}, \alpha)$.

If $F(\gamma) = 0$, where $\gamma = \mathcal{A}(\gamma)$, then we proceed exactly as in taking the ultrameasures other than those covered in Section 2 so we shall assume $F(\gamma) > 1$.

We have two basic problems. The first is to define iterated ultrameasures by ultrameasures $\delta(\gamma)$ on $\gamma$. Its second arises in taking ultrameasures by any $\delta(\gamma)$, but in only certain cases $\gamma = \mathcal{A}$. The problem is to define $\delta^*$ in the desired (iterated ultrameasures $\mathcal{M} = (M, \mathcal{A}, \alpha, \delta^*)$) set so that it is a $\tilde{\Psi}$ ultrameasure in $\mathcal{M}^*$. We shall discuss the first difficulty here, the second, which is closely related to the first, will be discussed later.

We first consider a simple ultrameasure $\mathcal{M} = (M, \mathcal{A}, \delta(\gamma))$. Let $\tilde{\Psi}^{(\gamma, \delta)}$ be the $\tilde{\Psi}$ ultrameasure of $(M, \mathcal{A}, \delta(\gamma))$, and that is $\tilde{\Psi}$ in the set of
equivalence classes \([E]^e\) where \(e \in E\). Assume \(E\) is a \(E\)-ultralimit sequence \((E^n)\) is well-defined. We will assume that \(E\) is well-founded and identify \(E^n\) with its transitive collapse.

We would like to define \(E\) so that it is a \(E\)-ultralimit sequence on \(p^e \in (E^n)\), but this is impossible: since every member of \(E\) is \(E\)-ultralimit in \((E^n)\), there exists \(\nu \in U\) such that every \(\nu \in \nu\) of \(p^e\) greater than \(\nu\) is \(E\)-ultralimit from parameters smaller than \(\nu\). If \(f\) is a \(E\)-function mapping each ordinal \(\alpha\) to the smallest parameter from which it is defined, then \(f\) is regressive except as a branch set, but \(f\) cannot be constant anywhere. Hence there is no \(E\)-ultralimit \(p^e\) of \(E\)-ultralimit on \(p^e\) in \((E^n)\).

Since we cannot construct any ultralimits on the ultrapowers \(E^n\), we cannot construct any \(E\)-ultralimits in the usual way as \(\{\nu \in \nu\}^E\). However it is possible to define the product ultrapower \(E^n \times E^n\) directly, and this is what we do. There are two cases, depending on whether \((p^e)\) is a successor or a limit ordinal.

The successor case \(\nu + 1\) is similar to the case of \(E\) - codes when \(p^e = e\) and will only require brief handling here. We will use the fact that in this case \(p\) is the least ordinal having a new subset definable in \(E\), and that every such subset has the form

\[\{\nu \in \nu \mid \exists \psi \in \nu (\psi \notin \psi)\}\]

for some \(E\)-set \(\nu\). We will show in Lemma 4.4a that \(E^n\) is a \(E\)-ultralimit. This enables us to define the map.
where $I$ is an iterated ultrapower (taking measures from $E$ on ordinals below the range of $p$) which generates a rank $\omega$ complete sequence of indiscernibles for interval $A \cup \{i(p)\}$. Then we will find that it is possible to choose $D' \preceq I_{p}(A \cup \{i(p)\})$. Thus we will find that it is possible to choose $D' \preceq I_{p}(A \cup \{i(p)\})$ such that $\gamma \equiv I_{p}(A \cup \{i(p)\})$ where $\gamma$ is the sequence of indiscernibles introduced by $I$.

We will solve the second problem by simply setting $F' = F$.

Since the fact that $\beta$ has an uncountable subset of ordinals $\beta' < \beta$ means (it can't be a compact sequence in) $\beta' \subseteq \beta$ and $\beta' < \beta$. This further, since our method of handling the second problem relies on $\beta$ being a limit ordinal.

One sequence that $\eta'(\gamma)$ is a limit ordinal. There are several steps involved in taking the ultrapower $I: E \rightarrow E'$ by $I_{p}(A \cup \{i(p)\})$. We will first take some ultrapowers by $I_{p}(A \cup \{i(p)\})$ to generate a cofinal sequence of indiscernibles and then we are the same way we can see a further ordinary iterated ultrapower is (iii) a complete center indiscernible. In the ultrapower $G'(\gamma)$ will be defined for every $\gamma' \in \gamma$ and for $\gamma \in \gamma'$ and $\lambda \in \lambda' \subseteq \gamma'$, as we do the final step in to set $G'(\gamma) = \gamma'$ where we do this recursion. We will see that every arbitrary choice of $\gamma'$ (iii) time wins.

In order to define the iterated ultrapower $I$ we first define a strictly increasing sequence $(A_{1}, \ldots, A_{n})$ cofinal in $\mathcal{P}(E)$, such that $A_{1} \preceq A_{2}$, and then the iterated ultrapower $I: (A_{1}, \ldots, A_{n}) \rightarrow I_{p}(A_{1}, \ldots, A_{n})$ using $F_{p}$ functions and the measures $(A_{1}, \ldots, A_{n})$. We will put off until later
the investigation of the property required of & in order for such an
interval $\mathcal{B}$ alphabet to make sense.

Thus $i_2$ in $i_1$-continuity, so $(\mathcal{B}, B_i)$ is simply reasonable and by
proposition 6.1 $i_1$ extends to a map of $i_1$-continuity regularizing
$(\mathcal{B}, B_i)$ to that regularizing $(\mathcal{B}, B_i)$. In particular, if $p = \text{ord}(i_1)$ then
$p_2 = i_2 \cdot i_1 \cdot \text{ord}(i_1)$. The sequence $g_2 = (i_2)$ is, below $p_2$, an
ultrafilter sequence (not an $\mathcal{A}$, ultrafilter sequence) on $\mathcal{B}$.

Although we don't define $g_2$ at $p_2$, it will be useful to establish
some notation so we move. Thus we set $\bar{A}^i = \{a \in B_i \mid l < \text{ord}(i)$,
noting that it may be smaller than $\bar{A}^i = \{a \in B_i \mid l < \text{ord}(i)$.

Now define $s_1$ to be $\bar{A}^i_s$ for $s < i$. Then $s_1$ can be regarded
as an admissible for $B_i$, $B_i$, $B_i$, and $i_2 = s_1 < i$. If $i_2$ is initial in
$B_i$, this sequence cannot be used to define $g_2$ as an ultrafilter sequence
on $B_i$, where each $B_i$ has at most one admissible, rather than a
ultrafilter sequence.

Now let $\mathcal{G} : B_i \to B_i$ be an ordinary honest ultrafilter by $B_i$, which
generates a complete system $\mathcal{C}$ of admissible for $B_i$ in the
open interval $(p, \infty)$ with $p = \text{ord}(i)$ and if $i$ is in the
extension of $i_2$ given by proposition 6.10. Then $C \subseteq B_i$ is $B_i$
ultrafilter sequence below $p$. Now set $\mathcal{A}^i = \{a \in B_i \}$, the set of
$\mathcal{A}^i$ defined in $B_i$ to extend $C$ to $p_2$ and then use this
extension to define $g_2$ at $p_2$. If we set $i = i_2$, then we define $\text{d}(x)_i$
for $x < \mathcal{A}^i$ by putting $x = \text{d}(x)_i$ iff there is some $i < i$ such that
either $\mathcal{A} = \text{ord}(i)$ and $x = u_i < \mathcal{A}$ or $x \in \mathcal{G}(\mathcal{A}) \setminus \text{ord}(i)_i$.

Such $\text{d}(x)_i$ is initial in $p_2$ and so $p_2$ can be used as a sequence.
of indistinguishable to define $\phi^+_{p'}$. Then

$$i = \langle \phi, p, \rho_0 \rangle \rightarrow (\phi, p^+)$$

is the desired ultrapower of $\mathcal{M}$ by $\varphi(p)$. Note that if desired we could
make $\phi$ much simpler below $p'$ rather than merely complete. In this case $\phi$
will be also an ultrametric at $p'$. The case of an ultrapower of $\mathcal{M}$ by $\varphi(p)$,
where $p, p'$ are simpler. Let

$$i : (\phi, p) \rightarrow (\phi, p^+) \in \mathcal{M}(p_0, p),$$

Then $(\phi, p^+)$ is uniquely $\phi$-equivalent and $i$ extends to $i : \mathcal{M}(p_0, p) \rightarrow \mathcal{M}(p_0, p^+)$.

If $\varphi(p, p')$ is a $S_1$-ultrametric then $p$ is in $\mathcal{M}$ regular in $\mathcal{M}(p_0, p)$ and
$(\phi(p, p'), \varphi(p, p'))$ can be uniquely extended to a $S_1$-ultrametric in $\mathcal{M}(p_0, p)$.

We now turn to the problem which we defined earlier of making sense of an
invariant $S_1$ ultrapower. Since invariant ultrapowers are based on finite
supports, the problem reduces to that of making sense of a finite invariant
ultrapower. For simplicity we will consider an invariant ultrapower of length
$\mathcal{M}$: if $\phi \subseteq \varphi, \psi \in \mathcal{M}$ then we will investigate the property required to make
$\varphi(p, p') \in \mathcal{M}(p_0, p)$ a $S_1$ ultrametric. Suppose $\mathcal{M}$ is a $S_1$
ultrametric of $\phi$ and $\varphi$ is defined by

$$\varphi(p, p') \subseteq \varphi(p, p'), \varphi(p, p') \in \mathcal{M}(p_0, p),$$

We need to show that either $\varphi(p, p') \in \mathcal{M}(p_0, p)$ or $\varphi(p, p') \in \mathcal{M}(p_0, p)$. Since $\mathcal{M}$ is
not a $S_1$ ultrametric sequence, it is not clear that either possibility need
hold. We observe, however, that if the ordinary commutative rules of
ultiproper bold that $x \in \mathbb{G}(x,j) \iff x \notin Q(x,j)$, where

$$x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{G}(x,j, y, z, a).$$

Here $G$ is a $\Sigma$-ultrapower sequence. $x_1, x_2, x_3, x_4, x_5$ are as $x \in S$. Thus either $x$ or its complement is in $\mathbb{G}(x,j)$, and $x \notin Q(x,j)$ should be decided accordingly.

A similar occurrence is the case $x \in p$. We have taken the ultrapower $\mathbb{G}(p, q, r, s, t) \to \mathbb{G}(p, q, r, s, t)$ and want to define $G(p, q, r, s, t)$ for $x \in \mathbb{G}(p, q, r, s, t)$. If $S$ is in $\Sigma_2$ and $|g| < \kappa$, then we expect to have $(x, \mathbb{H}(x, g)) \to (x, \mathbb{H}(x, g))$ if $x \in \mathbb{G}(p, q, r, s, t)$, where

$$x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{G}(p, q, r, s, t).$$

Again, $A$ is a set $\Sigma_2$, so constructively implies that $x \in \mathbb{G}(p, q, r, s, t)$ if and only if $x \in \mathbb{G}(p, q, r, s, t)$, where

$$x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{G}(p, q, r, s, t).$$

Thus, in both cases we get the ultrapowers to compute. This requires a condition on $G$, which is, so far as we know, not implied by the hypotheses that $G$ is a $\Sigma$-ultrapower sequence.

A 3. Definition: A filter $\mathcal{F}$ on $\mathcal{P}$ is non-trivial on $\mathcal{P}$ if for $A, B, C \subseteq \mathcal{P}$, where

$$A, B = \{a, b \in \mathcal{P} : (a, b) \in \mathcal{F}\} \cap \mathcal{P}$$

and $C = \{c \in \mathcal{P} : (c, d) \in \mathcal{F}\} \cap \mathcal{P}$, the sequence $\mathcal{F}$ is $\Sigma_2$-non-trivial in $\mathcal{P}$ if each $(A, B, C)$ is non-trivial on the class of sets which are $\Sigma_2$-closed.

The following lemma, showing that ultrapowers commute, is proved as usual using the fact that $\mathcal{F}$ is non-trivial. This is the generalization of Proposition 2.6 to $\Sigma_2$ ultrapowers.
4.20 Proposition: If $U$ is commutative on $s$ then:

(i) If $x \in p$, $R \in x \in p$ is in $F$, and $p$ is any filter on $x$ then $x \in \{ \langle a, b \rangle \in p \mid \langle a, b \rangle \in F \} \in c$ IFF $\langle a, b \rangle \in p \in \{ \langle a, b \rangle \in F \} \notin F$.

(ii) If $s$ is a filter on $x$ and $R \subseteq s$, any $F = \{ F \}$ then:

$\langle a, b \rangle \in \{ \langle a, b \rangle \in F \mid \langle a, b \rangle \in \{ \langle a, b \rangle \in F \} \in c \}.$

Proposition 4.22(1) is used in the case $a \neq p$, and Proposition 4.23(1) is used for the case $a = p$. This completes the definition of the ultrapowers; putting everything together we have the following lemma:

4.22 Lemma: Suppose that $U$ is a $\mathbb{N}$ ultrapower sequence and is $t_1$
commutative for $s$. Then $s \in \mathcal{D}(U)$ and that of $a \neq p$ that $\mathcal{D}(U)$ is a
limit ordinal. Then the ultrapower $(X \triangleleft X')$ of $X$ by $\mathcal{D}(U)$ is a $\mathbb{N}$
commutative embedding satisfying the following two properties: If $X$ is a
$\mathbb{D}$ formula then for all $|a| \leq s$, if $T$ is in the support for the limited
ultrapower then:

(i) $s \in \mathcal{D}(U)$ IFF $s \in \mathcal{D}(U) \cap \mathcal{D}(T)$

(ii) $s \in \mathcal{D}(U)$ IFF $\exists s \in \mathcal{D}(U) \cap \mathcal{D}(T)$. IFF $\exists t \in \mathcal{D}(U) \cap \mathcal{D}(T)$.

Another application of $\mathbb{N}$ commutativity is lemma 4.26 below, which
essentially says that if $s \in \mathcal{D}$ is a $\mathbb{N}$ ultrapower sequence for $X$ then there
is only one ordinal $\mathcal{Y} < \mathcal{D}(U)$ such that there is a new $\mathbb{D}$ subset of $s$
model $\mathcal{Y}$.

4.26 Definition: The reflectivity of $U$ is written $\mathcal{R}(U)$.
(a) Suppose \( p \neq 0 \) and \( \forall x \in X \), \( f(x) = p \).
(b) Suppose \( p > 0 \) and \( \forall x \in X \), \( f(x) = 0 \).
(c) Suppose \( p = 0 \) and \( \forall x \in X \), \( f(x) = a \) is a limit (null) sequence in \( \mathbb{R}^n \).

2.3 Lemma: Suppose \( d \in \mathbb{Z} \) ultralimit sequence for \( e \) and \( \beta \) is the ultralimit of \( d \). If \( \beta \) is a \( \mathbb{Z} \) formula then we may

(a) \( \forall x, \beta(x) \) is the \( \mathbb{N} \) formula \( \forall x \beta(x) \) in \( \mathbb{N} \), \( \forall x \beta(x) \) in \( \mathbb{Q} \), \( \forall x \beta(x) \) in \( \mathbb{R}^n \), \( \forall x \beta(x) \) in \( \mathbb{C} \).

(b) \( \forall x \beta(x) \) is \( \mathbb{N} \) formula.

Proof: First we suppose \( \forall x \beta(x) \) is \( \mathbb{N} \) formula. Let \( \forall x \beta(x) \) be a limit (null) sequence in \( \mathbb{N} \), \( \forall x \beta(x) \) is \( \mathbb{N} \) formula. We will show that the function \( \forall x \beta(x) \) defined by (1) is also \( \mathbb{N} \) formula. The proof for (2) is almost identical.

Let \( \gamma \) be a function taking a subset of \( \mathbb{N} \) formula into \( \mathbb{N} \). Fix an \( \varepsilon \), \( \delta \) and define a \( \mathbb{N} \) function \( g_{\varepsilon, \delta}(n) \in \mathbb{N} \) so that if \( \forall x \beta(x) \) then \( \exists \varepsilon, \delta \) is bounded by \( \varepsilon \), since \( \beta \) is \( \mathbb{N} \) formula. Then \( g_{\varepsilon, \delta}(n) \) is \( \mathbb{N} \) formula. Thus \( \forall x \beta(x) \) in \( \mathbb{N} \) formula.

This completes the case \( \forall x \beta(x) \in \mathbb{N} \).
The proof of Lemma 4.11 also proves the following corollary, which says essentially that, modulo $\mathfrak{M}$, the only new $\exists \mathbb{Z}$ formulas are those whose domain is $\forall \mathbb{Z}$.

**Corollary:** If $\mathfrak{M}$ is in $\exists \mathbb{Z}$ then ($\exists \mathbb{Z}$) every $\exists \mathbb{Z}$ function $g$ with
domain \( K = \mathbb{C} \) is a number of \( K \), and (iii) if
\[
[\text{char } K = p, \alpha = \alpha_{[x]}, \text{ and } \omega \text{ is a unimodular } O \text{-function } \mathcal{O}(x,z),
\]
then there is \( h \in \mathcal{O}(x,z), \) such that \( h = \omega'. 
\]

Weak \( H \) codes: Suppose \( G = \langle H, \lambda \rangle \) is a unimodular and \( H \)-complete unimodular graph \( G \) in a \( \mathcal{H} \) sequence. Suppose \( G \) is a \( \mathcal{H} \) sequence. Suppose \( G \) is the first step towards defining the \( \mathcal{H} \) code of \( G \). which will be a complete, all of the information definable in \( G \) from \( G \), we define the weak \( \mathcal{H} \) code which is the \( \mathcal{H} \) relation but ignores all further information in \( G \). The basic tool is a reduction of \( \mathcal{H} \).

A set \( \mathcal{H} \) is reducible in the usual \( \mathcal{H} \) sense to the parameter \( x \) via the parameter \( \omega \) if \( (\omega, x) \in \mathcal{H} \), and there is a function \( \mathcal{H} \) of \( \mathcal{H} \) definable for \( x \) over \( (\omega, x) \) such that \( \mathcal{H} = \mathcal{H}(\tilde{x}, \bar{x}) \), where \( \tilde{x} \in \mathcal{H}(x, \bar{x}) \). The reduction is rank \( \mathcal{H} \) if the system \( \mathcal{H} \) is rank \( \mathcal{H} \). complete above \( \omega \).

It should be observed that we are implicitly identifying \( \omega \) with \( (x,x) \) here. A more precise sense would replace \( \omega \) with \( (x,x)^{\omega} \)
throughout.

A reduction is said to be canonical if \( \omega \) is as small as possible and the parameter \( y \) is as small as possible for this \( \omega \).

A set \( \mathcal{H} \) is \( (x,y) \) \( \mathcal{H} \), or \( \mathcal{H} \) is \( (x,y) \) \( \mathcal{H} \) if \( \mathcal{H} \) is \( \mathcal{H} \) unimodular and there is an \( \omega \) \( \mathcal{H} \) and parameter \( \omega \) such that \( \mathcal{H} \) is \( \mathcal{H} \) rank \( \mathcal{H} \) complete reducible to \( \omega \) via \( \omega \) and \( \mathcal{H} \) \( \mathcal{H} \) definable in \( \mathcal{H} \) from \( \mathcal{H} \).
The task Ω will actually make up the truth of \( \Sigma \) sentences involving arbitrary parameters from \( \delta \), not just parameters from \( \mu \cup \eta \). Suppose \( \sigma \) is \( \Sigma \) and \( \rho \in U \). Then \( \sigma \) can be written in the form \( f(\zeta) \) where \( f \) is \( \in \Sigma \) in parameters from \( \mu \cup \eta \). all \( i \) is equivalent to \( \mu(\zeta(i)) \), which in \( \Sigma \) is the sequence \( \mathcal{E} \) of indiscernibles. The sequent \( \mathcal{E} \) is a \( \varepsilon \)-match with \( \varepsilon \)-sequence \( \mathcal{E} \) has sequence

\[ G = \left( \eta_{0}(\zeta), \ldots, \eta_{n}(\zeta) \right) \]  

Then each of the functions \( \eta_{0} \) and \( \eta_{1} \) is \( \Sigma \) with parameters from \( \mu \cup \eta \) and hence can be written in the form \( f_{i}(\zeta) = k_{i}^{\mu}(\zeta, \eta_{i}(\zeta)) \) and \( g_{i}(\zeta) = k_{i}^{\eta}(\zeta, \eta_{i}(\zeta)) \) for some \( k_{i} \) and \( g_{i} \) \( \in \mu \). We will say that the ordinal coding the sequence \( \eta_{0}(\zeta), \ldots, \eta_{n}(\zeta) \) codes the support of \( \mathcal{E} \).

A.26 Proposition: For any \( \Sigma \) function \( \delta \) there is an indiscernible function \( \xi \) that for all ordinals \( \eta_{0}, \eta_{1} < \delta \), if \( \eta_{0} \) codes the support of \( \mathcal{E} \) then \( \delta \models \text{Wide}_{\mathcal{E}} \) \( \eta_{0} \) \( \models \text{Wide}_{\mathcal{E}} \).

Proof: Let \( \Gamma \) be the support for \( \mathcal{E} \) coded by \( \eta_{0} \). Then \( \delta \models \text{Wide}_{\mathcal{E}} \) if and only if

\[ \text{there exists } \zeta \in 2 \]  

such that \( \delta \models \text{Wide}_{\mathcal{E}}(\zeta) \) (or \( \models \text{Wide}_{\mathcal{E}}(\zeta) \)).

We will generally refer to \( \text{Wide}_{\mathcal{E}}(\zeta) \) for the ordinal coding of \( \mathcal{E} \) and \( \xi \) given by Proposition 4.26. Then 4.26 could be rewritten as for all \( \zeta, \xi \) \( \delta \models \text{Wide}_{\mathcal{E}}(\zeta) \) \( \delta \models \text{Wide}_{\mathcal{E}}(\zeta) \) \( \delta \models \text{Wide}_{\mathcal{E}}(\zeta) \). Notice that
since ω((κ, ω)) is rudimentary it is bounded by a rudimentary function in ξ, that is, ω((κ, ω)) is a rudimentary function of μ and α for the support of ξ.

4.28 Lemma. Suppose N = (N, E) is a weak $\mathbb{B}$-code of (B = (B, A, E)). Then

(i) $\mathfrak{A}[\mathfrak{A}]$ and $\mathfrak{B}$ is the least ordinal having a $\mathfrak{B}$-subset which is not a member of $\mathfrak{B}$.

(ii) If $E = (E, E')$ is another weak $\mathbb{B}$-code for $\mathfrak{B}$ then $\mathfrak{B} \neq \mathfrak{B}$ and $\mathfrak{B}$ and $\mathfrak{B}'$ are each rudimentary in the other. If the parameter used to obtain $\mathfrak{B}'$ is the same as that used to obtain $\mathfrak{B}$, then $\mathfrak{B} \neq \mathfrak{B}'$.

Proof. If $\mathfrak{B} = \mathfrak{B}'$ then by Corollary 4.25 and Proposition 4.19 as in

$x \in \omega((x, x)) \in x$.

(i) $\mathfrak{B}$ is the support for $\mathfrak{B}$ then $\mathfrak{B}(x, x)$ exists and $\omega((x, x)) \neq \mathfrak{B}(x, x)$. But then $\mathfrak{B} = \mathfrak{B}(x, x)$ for some $\mathfrak{B} = \mathfrak{B}(x, x)$ with support $\mathfrak{B}$, so $\mathfrak{B}(x, x) \neq x$ and $\mathfrak{B}(x, x) \neq \mathfrak{B}(x, x)$. Hence $\mathfrak{B}$ cannot be a member of $\mathfrak{B}$.

Now suppose $\mathfrak{A} \neq \mathfrak{B}$ and $\mathfrak{B}$ is a $\mathfrak{B}$-subset of $\mathfrak{A}$. Say $\mathfrak{A} = \omega((x, x)) = \mathfrak{B}$. Then $\mathfrak{A} = (\mathfrak{A}, \mathfrak{A}) \neq \omega((x, x)) = \mathfrak{B}$. This is a $\mathfrak{B}$-rudimentary function of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A}$ is a $\mathfrak{B}$-code for the support of $\mathfrak{B}$.

Since $\mathfrak{B}$ is bi-symmetric, it is bi-symmetrically closed and as $\mathfrak{B} \neq \mathfrak{A}$, $\mathfrak{B}$ is not a member of $\mathfrak{B}$.

If $\mathfrak{B}$ is another $\mathfrak{B}$-code then $\mathfrak{B} \neq \mathfrak{B}$ by (ii) and each is rudimentary in the other by 4.19. We must have $\mathfrak{B} = \mathfrak{B}(x, x)$, where $\mathfrak{B}$ is a $\mathfrak{B}$-code for the support of $\mathfrak{B}$, so
\( b = b' \). Observe that \( \beta_e(x) \) does not depend on what \( d \) is above \( e \), although it is unchanged by inverted diagrams. Finally, if \( e = e' \) then \( b = b' \) by Proposition 2.3b and hence \( b = b' \) by Definition 4.9b.

**Decoding:** A decoding of a structure \( B \) is a structure \( B' \) such that \( B \) is the weak \( L^m \) code of \( B' \). We will be concerned with three questions concerning decodings of a structure \( B' \).

1. When \( B \) is a decoding, what does the decoding look like, and how does an embedding \( \beta_e \) extend to an embedding between the decodings \( B' \) of \( B \) and \( B'' \)?

We deal with the second question first. Corollary 4.12 cannot be reversed; the decoding is not determined uniquely by its weak \( L^m \) code \( B' \). This is true in the first place because the weak code does not completely determine the sequence of arrows in \( G \) (the full \( S^n \) code is needed for this) but more importantly it is true because \( B' \) does not determine the sequence \( C \) of indecomposables used in the reduction of \( B \). In fact, we have

**4.12 proposition:** Suppose that \( B \) is the weak \( L^m \) code of \( B' \) and \( C \) is a thrombolat diagram such that \( \text{span}(B) \) is full. Then \( B' \) is also a weak \( L^m \) code of \( B' \).

**Proof:** The reduction of \( B \) to \( B' \) is a reduction of \( B' \) to \( B' \) using the indecomposables from the reduction of \( B \) together with those added by the thrombolat diagram \( C \).

Warning: For this the decoding of \( B' \) is unique. In particular there is a unique decoding \( B \) of \( B' \) such that the reduction of \( B \) to \( B' \) does not require
Every model \( M \) of \( \mathcal{L} \) has an innermost instance \( M' \). For every \( \psi_0 \), \( \psi_1 \), \( \psi_2 \), \( \ldots \) of \( \mathcal{L} \), there exists a function \( f : M \to M' \) such that for all \( \phi \) of \( \mathcal{L} \):

\[
\phi(v_1, \ldots, v_n) \iff \exists x, y_1, \ldots, y_m. \phi^f(x, y_1, \ldots, y_m).
\]

\( \phi^f(x, y) \) is in the innermost instance of \( \phi \) whenever \( \phi^f(x, y') \) is in the innermost instance of \( \phi \) whenever \( \phi^f(x, y) \) is in the innermost instance of \( \phi \), and \( \phi^f(x, y) \) is in the innermost instance of \( \phi \) whenever \( \phi^f(x, y) \) is in the innermost instance of \( \phi \), and \( \phi^f(x, y) \) is in the innermost instance of \( \phi \) whenever \( \phi^f(x, y) \) is in the innermost instance of \( \phi \).
An important special case occurs when the classical ultrapower \( U = \{ p \} \) is a formula defining a function \( \pi \). Then an ultrapower will actually determine the value of \( x \) in the decoding \( \pi \) of \( \pi \). If \( \pi \) takes its maximum possible value if there is no interpretation on \( \pi \), and \( \pi \) takes its minimal possible value if there is no interpretation on \( \pi \), then \( \pi \) will take the smaller value if \( \pi \) includes the sentence \( \phi \). This fact will be important in dealing with daughter \( \pi \)'s of \( \pi \) discussed earlier. Suppose that \( \phi \) is itself an iterated ultrapower of some structure \( \pi \) and recall from \( \pi \) that in some cases the definition of the ultrapower specified a reduction of the value of \( \pi \) by its natural value. This reduction will require the use of an ultrapower \( \pi \), \( \pi \) of \( \pi \) in the construction of the decoding \( \pi \). For a similar reduction of the value of \( \pi \) by its natural value, see \( \pi \). This issue only arises when \( \pi \) is a full \( \pi \) code, since the weak \( \pi \) codes do not include any information about \( \pi \) or \( \pi \).

We now turn to the first question: when does a "reasonable" structure \( \pi \) have a decoding \( \pi \)? The construction of the core coding \( \pi \) is straightforward. We take \( \pi \) to be the set of "equivalence classes of pairs \((x, y)\) where \( x \) is a \( \pi \) formula defining a function \( x \) and \( y \) a \( \pi \) and the equivalence relation \( \sim \) is defined by
\[
\pi \sim (x, y) \iff (x') \pi (y') \quad \text{and} \quad (y') \pi (x')
\]
The formula \( \pi \left( y' \right) \) is \( \sim \) in \( \pi \) is determined by \( \pi \):
\[
\pi \left( y' \right) \sim (x', y') \quad \text{and} \quad \pi \left( x' \right) = \pi \left( y' \right)
\]

The statement asserting that this makes sense, that "it is so"
equivalence relation so that a sentence \\((\phi, a)\) is exactly "\(B\)-valid" in \(F_{\mathfrak{a}}\) - well-founded.

The \(T_\mathfrak{a}\) part reduces to a \(T\)-sentence in \(B\), so it will be true whenever \(B\) is
\(\mathcal{E}_B\) elementarily equivalent to any weak base. as in lemma 4.10, \(B\) will be well
founded whenever \(B\) is either a submodel of a weak base or an iterated
ultrafilter of a weak base by a consistently complete ultrafilter.

The "\(B\)-validity of \(B\) means that \(B\) is known except for its domain. But
its domain is given by the \(x_{\mathfrak{a}}\) relation

\[\mathcal{E}_B\ (\mathcal{E}_B, \mathcal{B}) \quad 1 \quad \mathcal{E}_B \psi B, \psi \in \mathcal{B} \]

Finally \(\mathcal{E}_B\) can be recovered from \(\mathcal{E}_B\). If \(\psi\) is a \(\mathcal{E}_B\) formula then
\[\mathcal{E}_B \psi B, \psi \in \mathcal{B} \psi \in \mathcal{B} \mathcal{B} \]

This completes the definition of the \(\mathcal{E}_B\) encoding \(\mathcal{E}_B\) of \(\mathcal{B}\) and we have proved:

4.10 Lemma: Suppose that \(B\) is the weak \(\mathcal{E}_B\) code of \(\mathcal{B}\), that \(B\) \(\mathcal{E}_B\) \(\mathcal{B}\), and \(B\) is
"\(\mathcal{E}_B\)-valid. Then the proof by description above builds produces a structure \(\mathcal{E}_B\), and
if \(\mathcal{E}_B\) and all its iterated ultrafilters are well founded then it is a weak \(\mathcal{E}_B\)
encoding of \(\mathcal{B}\).

4.10 Lemma: Suppose that \(B\) is the weak \(\mathcal{E}_B\) code of \(\mathcal{B}\) and either (i)
\[\phi \phi B, \phi \in \mathcal{B} \quad \mathcal{E}_B \psi B, \psi \in \mathcal{B} \mathcal{B} \]
or (ii) \(\mathcal{B}\) is an iterated ultrafilter by consistently
complete ultrafilters. Then \(B\) has a weak \(\mathcal{E}_B\) encoding \(B\) such that \(B\) can be
extended in a way between \(\mathcal{B}\) and \(\mathcal{B}\).
Proof: We first observe that in either case all of the iterated ultrapowers of the inverse embedding \( \mathcal{N}_0 \) of \( \mathcal{M} \) are well founded. In case (ii) every iterated ultrapower can be embedded into the corresponding iterated ultrapower of \( \mathcal{M} \) which is well founded by assumption, while in case (iii) standard arguments imply that \( \mathcal{N}_0 \) and all of its iterated ultrapowers are well founded.

In case (ii) we can immediately extend the embedding \( \mathcal{N}_0 \) to a map \( \mathcal{N}_0 \rightarrow \mathcal{M} \) and we can take \( \mathcal{N}_0 \) to be \( \mathcal{N}_0 \). It should be noted, however, that we can also take \( \mathcal{N}_0 \) to be an iterated ultrapower of \( \mathcal{N}_0 \) provided that the corresponding ultrapowers exist in \( \mathcal{M} \) so that the map can be extended. In particular we can insist that \( \mathcal{N}_0 \) be rank \( \omega \) (or greater) complete, provided that \( \mathcal{M} \) also is.

In case (iii) \( \mathcal{N}_0 \) will generally have to be taken to be an iterated ultrapower of \( \mathcal{N}_0 \) so that it has indiscernibles into which to map the indiscernibles of \( \mathcal{M} \). The map may be constructed by starting with \( \mathcal{N}_0 \), the inverse embedding of \( \mathcal{M} \), and constructing a map \( \mathcal{N}_0 \rightarrow \mathcal{M} \). Then \( \mathcal{N}_0 \) is an iterated ultrapower of \( \mathcal{N}_0 \) and \( \mathcal{M} \) generates a map \( \mathcal{N}_0 \rightarrow \mathcal{M} \) to the corresponding iterated ultrapower of \( \mathcal{N}_0 \). Of course this process will not be affected if we add more indiscernibles to \( \mathcal{M} \) that are necessary for the embedding, so we can insist that \( \mathcal{N}_0 \) be rank \( \omega \) complete, if desired.

Note that in cases (ii) and (ii) the only way we make of the fact that \( \mathcal{M} \) is an iterated ultrapower by countable complete ultrapowers was in showing that \( \mathcal{M} \) is countable and that \( \mathcal{N}_0 \) and its iterated ultrapowers are well founded. In section 7 we will be using a slightly more general hypothesis for the same purpose.
We will make extensive use of the fact that the embedding between the dendrons $B$ and $B'$ is stronger than the original embedding between $B$ and $B'$, due to the information coded in the set $H$ of $B$. For the case when $B'$ and $B$ are isomorphic, the relation in the same as in $L$, the embedding is one that preserves stronger.

**Lemma:** Suppose that $l: B \rightarrow B'$, $r: B \rightarrow B'$, and $s: B \rightarrow B'$, where $l$ and $r$ are dendrons of $B$ and $B'$, and suppose that $l'$ is related to the same that every dendron in the induction of $B'$ to $B$ in the same manner $r'$ of some isomorphic in the induction of $B$ to $B'$. Thus $l'$ is $l'$ of $l'$ elementary.

**Proof:** Let $B \rightarrow B$ be the dendron of $B$. Then $B \rightarrow B$ is a dendron of $B$. Then $B \rightarrow B$ is a dendron of $B$. Then $B \rightarrow B$ is a dendron of $B$. Then $B \rightarrow B$ is a dendron of $B$. Then $B \rightarrow B$ is a dendron of $B$. Then $B \rightarrow B$ is a dendron of $B$.

This proposition can be extended somewhat—e.g., if $l'(l)(l') = l(l)$ and both $l$ and $l'$ have such a complete system of isomorphisms above $l(l')$ and $l(l')$, then the conclusion holds. The existence of the proposition is not always true, however. Suppose, for example, that $p = l(l')$ and that there is a total normal $l'$ function $(p \rightarrow l(l'))$. Let $b: B \rightarrow B'$ be the identity and $l': B \rightarrow B'$. Then for $B \rightarrow B'$, $p \rightarrow l(l')$ and $(p \rightarrow l(l')) = B'$. The failure of proposition 4.11 to generalize becomes important because...
as we have seen, an iterated ultraproduct $\hat{\mathbb{B}} \longrightarrow \mathbb{B}$ of the full $\mathbb{B}^*$ code $\mathbb{B}$ may require that the extension $\hat{\mathbb{B}} \longrightarrow \mathbb{B}$ include an ultraproduct of $\mathbb{B}^*$, in order to properly adjust $\mathbb{B}$.

These properties $\mathbb{B}$ cannot be applied directly to the extension $\hat{\mathbb{B}}$ of $\mathbb{B}$. The solution is to break the extension into two parts

$$1^* \mathbb{B} \longrightarrow 1^* \mathbb{B} \longrightarrow \mathbb{B} \longrightarrow \mathbb{B},$$

so that $1^*$ and $\mathbb{B}^*$ are developments of $\mathbb{B}^*$. We take $\hat{\mathbb{B}}^*$ to be the smallest extension of $\mathbb{B}$, so that proposition 4.28 applies, and we take $\mathbb{B}^*$ to be an iterated ultraproduct.

This process is a typical argument and we shall observe that the properties we want to be preserved are $\mathbb{B}^*(\mathbb{B}^*)$, so that they are preserved by $\hat{\mathbb{B}}$, and that they are preserved by $\hat{\mathbb{B}}^*$ because iterated ultraproducts are, in their own way, also well behaved. $\mathbb{B}^*$ is only a full $\mathbb{B}^*$ code of the final structure $\mathbb{B}$, out of the intermediate structure $\mathbb{B}^*$, but this is not a problem.

Full $\mathbb{B}^*$ and $\mathbb{B}^*$ codes: The construction of the full $\mathbb{B}^*$ code of $\mathbb{B}$ is straightforward: $\mathbb{B} = (G,A,A(x)\mathbb{B})$. Here, the weak $\mathbb{B}^*$ code $\mathbb{B} = (G,A)$ is straightforward.

$$\mathbb{B} = (G,A,G_0 \mathbb{B})$$ where $G_0$ is the restriction of the regularizing function for $\mathbb{B}$. Lemma 4.5 extends to full $\mathbb{B}^*$ codes, but we shall show that properties 4.5 holds only in the weaker form. $1^* \mathbb{B} \longrightarrow 1^* \mathbb{B}$ extends to

$$1^* \mathbb{B} = 1^* \mathbb{B} \longrightarrow \mathbb{B}$$ where $\mathbb{B}$ and $\mathbb{B}^*$ are full $\mathbb{B}^*$ developments of $\mathbb{B}^*$ and $\mathbb{B}^*$, in $\mathbb{B}^*$, respectively, and $\hat{\mathbb{B}}$ is an iterated ultraproduct such that $1^* \mathbb{B} = 1^* \mathbb{B} \longrightarrow \mathbb{B}$.

The $\mathbb{B}^*$ code of a model $\mathbb{A}$ is defined by induction over $\mathbb{A}$. The $\mathbb{B}^*$ code has already defined. For $n \geq 1$, $\mathbb{A}$ is a $\mathbb{B}^*$ code if $\mathbb{A}$ is a $\mathbb{B}^*$ code of $\mathbb{A}$ if it is a $\mathbb{B}^*$ code of a $\mathbb{B}^*$ code. $\mathbb{B}$ is the canonical code if it is the $\mathbb{B}^*$ code.
will be no two subsets of \( x \) and \( y \) in \( \mathcal{D}_1 \), so \( \mathcal{D}_1 \) is an ultrafilter sequence for \( \mathcal{D}_1 \). The sequence \( \mathcal{D}_1 \) will be an ultrafilter sequence at \( \mathcal{D}_1 \) because any union of \( x \) in \( \mathcal{D}_1 \) is definable in some \( \mathcal{L}_2 \) proof of \( \mathcal{D}_2 \), and \( \mathcal{D}_2 \) will be an ultrafilter sequence above \( \mathcal{D}_1 \) because it is generated by a rank complete ultrafilter \( \mathcal{D}_2 \). Lemma 4.3(ii) follows from the fact that there are no interdefinable between \( x \) and \( x'' \). (ii) from 4.3a.

We prove 4.5 by induction with the same induction hypothesis as for 4.5, namely that 4.5 holds for \( \mathcal{D}_1 \) whenever \( \mathcal{D}_1 \) is a proper initial segment of \( \mathcal{D} \) and for \( \mathcal{D}_1 \) whenever \( \mathcal{D}_1 \). In addition we assume that 4.5 is true for \( x \) such that \( \mathcal{D}_1 \) has a canonical \( \mathcal{L}_2 \) code \( \mathcal{D}_1 \rightarrow (N_2, N_2) \) when \( \mathcal{D}_1 \rightarrow (N_2, N_2) \). We will show first that \( \mathcal{D}_1 \) is in a \( \mathcal{D}_1 \)-consistent \( \mathcal{L}_2 \) subsequence of \( \mathcal{D}_1 \) and then that the \( \mathcal{D}_1 \)-consistent \( \mathcal{L}_2 \) subsequence of \( \mathcal{D}_1 \) exists.

A special case arises in the case when \( \mathcal{D}_1 \rightarrow (N_2, N_2) \) is a successor ordinal, similar to the case of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) when \( \mathcal{D}_2 \) is equal to \( x \). In that case we will prove the induction step of Lemma 6.5 directly (see Lemma 6.5 and Corollary 6.5) rather than explicitly defining \( \mathcal{D}_1 \) codes for \( x \)'s. As in the \( \mathcal{D}_1 \) case, definition of \( \mathcal{D}_1 \) codes is implicit in the argument. We assume in what follows that this special case has not occurred for any \( x \).  

4.5 Lemma: \( \mathcal{D}_1 \) is a \( \mathcal{D}_1 \)-consistent \( \mathcal{L}_2 \) ultrafilter sequence for \( \mathcal{D}_1 \).

The proof of Lemma 6.5 will run over the next few lemmas. The first
two results are the basic tools in the proof.

4.20 Proposition: Suppose \( B = (A_i, \mathcal{F}(A_i)) \) is a \( \mathcal{F}_1 \) code, \( x \in A_i \), \( x \) is a \( \mathcal{F}_1 \) subset of \( A_i \) with parameters from \( x \cup y \), and \( C \) is a rank 0-complete system of indiscernibles in \( B \) for \( (x, y) \) where \( x \) over parameters from \( x \cup y \). Then \( x \) is a member of \( x \mathcal{F}_1 \cdot \mathcal{F}_1 \)-meas.

Proof: \( B \) can be encoded to \( \mathcal{F}_1 \) which has a rank 0-complete system \( C_0 \) of indiscernibles above \( x \). But then \( C_0 \cdot \mathcal{F}_1 \mathcal{F}_1 \) is a rank 0-complete system of indiscernibles for \( B \) above \( x \) and hence \( C \) is an indiscernible sequence above \( x \) in \( \mathcal{F}_1 \) but \( x \in \mathcal{F}_1 \mathcal{F}_1 \), thus \( x \) is definable in \( \mathcal{F}_1 \mathcal{F}_1 \), and \( \mathcal{F}_1 \mathcal{F}_1 \) can be collapsed to obtain the desired mean.

4.21 Proposition: Suppose \( B = (A_i, \mathcal{F}(A_i)) \) is a \( \mathcal{F}_2 \) code, \( x \in \mathcal{F}(A_i) \), and \( x \in \mathcal{F}_2 \mathcal{F}_2 \). Then there is a \( \mathcal{F}_2 \mathcal{F}_2 \)-measure \( \mathcal{F}_2 \) with \( (x, y) \in \mathcal{F}_2 \).

Proof: Let \( \mathcal{F}_2 \) be the \( \mathcal{F}_2 \) encoding of \( \mathcal{F}_2 \), or that \( \mathcal{F}_2 \) is a \( \mathcal{F}_2 \mathcal{F}_2 \) code, and let \( i: \mathcal{F}_2 \to \mathcal{F}_2 \) be the ultrapower of \( \mathcal{F}_2 \) by \( \mathcal{F}(A_i) \), so that \( \mathcal{F}_2 \) has a rank 0-complete system of indiscernibles above \( x \). Now \( \mathcal{F}_2 \) in still \( \mathcal{F}_2 \mathcal{F}_2 \) definable in \( \mathcal{F}_2 \) but \( \mathcal{F}_2 \) is a \( \mathcal{F}_2 \mathcal{F}_2 \) code by Proposition 4.11. Thus \( \mathcal{F}(A_i) \) is in a \( \mathcal{F}_2 \mathcal{F}_2 \mathcal{F}_2 \)-measure by Proposition 4.14.

4.22 Corollary: \( \mathcal{F}_2 \) is a \( \mathcal{F}_1 \) indiscernible sequence for \( \mathcal{F}_2 \).

Proof: We will show that if \( x \in \mathcal{F}_2 \mathcal{F}_2 \) is a \( \mathcal{F}_1 \mathcal{F}_2 \) subset of \( y \), then \( x \) is in a \( \mathcal{F}(x, y) \)-meas., and hence \( x \in \mathcal{F}(x, y) \). The corollary follows immediately by the assumption that \( \mathcal{F}(x, y) \) is a \( \mathcal{F}(x, y) \)-ultrafilter. 
If \( \alpha = \beta \), then the required lemma comes immediately from Proposition \( 4.20 \). If \( \alpha < \beta \), then let \( \alpha \downarrow \beta \) be an interval upper-semi-open if \( \alpha \downarrow \beta = \{ \gamma \mid \gamma \leq \alpha \leq \beta \} \) and \( \alpha \downarrow \beta \) is a rank-\( \omega \)-complete system of indiscernibles above \( \alpha \). \( \alpha \downarrow \beta \) is a \( \Sigma_2 \) code by Propositions \( 4.19 \) and \( \alpha \downarrow \beta \) is still \( \mathbb{E} \) in \( \mathcal{B} \). We will now take care of the successor case: assume \( \mathcal{T}(\alpha) = \lambda + 1 \)
and set \( \alpha = \nu_0 \) and \( \beta = \nu_1 \).

**Lemma 4.25**: Suppose \( \alpha = \beta \). \( \alpha \downarrow \beta \) is \( \mathbb{E} \) in \( \mathcal{B} \). and \( \beta \).

Note: \( \mathcal{P}(\alpha, \beta, \mathbb{E}) \) in \( \mathcal{B} \). Then \( \lambda \) is in \( \mathcal{B}(\mathbb{E}) \).

**Proof**: The lemma is proved by induction on \( \alpha \). For \( \alpha = \beta \) it follows from Proposition \( 4.18 \). Assume that \( \alpha \downarrow \beta \) and the lemma is true for \( \alpha - 1 \).

We will use the induction hypothesis and the fact that \( \mathcal{B} \) is a \( \mathcal{B}(\mathbb{E}) \)-ultrafilter we can take the ultrafilter:

\[
S_\alpha = S = (\alpha, \beta, \mathbb{E})
\]

This is \( \chi(x, j, \beta, \mathbb{E}) \) where \( \mathcal{E} = (\mathbb{E}, j, \beta, \mathbb{E}) \) and \( \mathcal{E} = (\mathbb{E}, \beta, \mathbb{E}) \).

For \( \alpha \downarrow \beta \) is in \( \mathcal{B} \). It follows from Proposition \( 4.17 \) that \( \gamma \) is in \( \mathcal{T}(\beta, \mathbb{E}) \)-ultrafilter and hence is in \( \mathcal{B}(\mathbb{E}) \).

**Corollary 4.26**: If \( \mathcal{T}(\alpha) = \lambda + 1 \) then every subset of \( \alpha^\prime \) in \( \mathcal{B} \) has

the form \( \{ \langle \omega \rangle \alpha \in \mathcal{A}(\mathbb{E}, \alpha) \} \) \( \mathbb{E} \} \) for some \( \mathbb{E} \) and some \( \mathcal{E} = (\mathbb{E}, \alpha, \mathbb{E}) \) subset \( \mathcal{E} \) of \( \mathcal{T}(\alpha) \). Hence \( \mathcal{T}(\alpha) = \alpha^\prime \) for all \( \mathbb{E} \) and common to all \( \alpha^\prime \) held at \( \omega \).
The first sentence is an easy induction using Proposition 1.6 and Lemma 4.6. The second sentence follows immediately by 4.4 and the assumption on $\mathcal{F}$.

This completes the proof of Lemma 4.19 and 4.20 for the case in which $\mathcal{P}(\lambda^+)$ is a limit ordinal. Throughout the rest of the proof of Lemma 4.20 (that is, the rest of Section 4) we will assume that $\mathcal{P}(\lambda^+)$ is either 0 or a limit ordinal. If $\mathcal{P}(\lambda^+)$ is then Lemma 4.20 follows immediately from Corollary 4.42, so for the rest of the proof of Lemma 4.20 we will also assume that $\mathcal{P}(\lambda^+)$ is 0.

Consider $\mathcal{P}(\lambda^+)$ is a limit ordinal, $\lambda \in \mathcal{P}(\lambda^+)$, and $\alpha \in \widehat{\lambda}$ then $(\omega, m_{\alpha}|\alpha)$ is in $\#\lambda_{\omega}^{\frac{\infty}{\infty}}$. Since $\mathcal{P}$ is $\mathcal{P}_\omega$-continuous in $\mathcal{P}_0$.

Proof: The first sentence is proved by taking the ultrafilter in $\mathcal{P}(\lambda^+)$ as in the case $\alpha = 1$ of Lemma 4.43. To prove the second sentence, first note that using the notation of Definition 4.20 if $\alpha \in \mathcal{P}_\omega \in \mathcal{P}_0$ then by the first sentence there is an $\mathcal{P}(\lambda^+)_{\mathcal{F}}$-name containing both $\mathcal{F}$ and $\mathcal{P}_0$. Then the function $F = \mathcal{P}_\omega \times \mathcal{P}_0$ is also in the sense of $\mathcal{F}$.

In the proof of Lemma 4.20 we will actually only need what has already been proven, that $\mathcal{F}$ is a $\mathcal{P}_\omega$-continuous, $\mathcal{P}_\omega$-ultrafilter sequence. The fact that $\mathcal{F}$ is a $\mathcal{P}_\omega$-ultrafilter sequence will fail not during the proof.
Proof: The first sentence is an easy induction using Proposition 4.1 and Lemma 4.3. The second sentence follows immediately by 4.43 and the assumption on $\mathcal{H}$.

This completes the proof of Lemmas 4.4 and 4.45 for the case in which $\mathcal{H}$ is a maximum element. Throughout the rest of the proof of Lemma 4.4 that is, the rest of Section 4— we will assume that $\mathcal{H}$ is either $\sigma$ or a limit ordinal. If $\mathcal{H} = \sigma$ then Lemma 4.4 follows immediately from Corollary 4.47, so we need the proof of Lemma 4.45. For the rest of the proof of Lemma 4.45 we will also assume that $\mathcal{H} = \sigma$.

**Lemma 4.45:** If $\mathcal{H} = \sigma$ is a limit ordinal, $\lambda < \mathcal{H}$, and $\mathfrak{B} = \mathcal{H}_\lambda$, then $\mathfrak{B} = \mathcal{H}$ is a $\mathcal{H}_\lambda$-sequence. Since $\mathfrak{B}$ is $\sigma$-cumulative in $\mathcal{H}$,

Proof: The first sentence is proved by taking the ultrapower by $\mathcal{H}_\lambda$ in the case $\lambda = \sigma$ of Lemma 4.45. To prove the second sentence, first note that taking the notation of Definition 4.43, if $\mathfrak{B} = \mathcal{H}_\lambda$, then by the first sentence there is a $\mathcal{H}_\lambda(\mathcal{H}_\lambda)$-sequence containing both $\mathfrak{B}$ and $\mathcal{H}$. Then the functor $\mathfrak{F} = \mathcal{H}(\mathcal{H}_\lambda)$ is also in the sequence. If $\mathfrak{M} = \mathfrak{B}$, then $\mathfrak{F}$ is isomorphic and $\mathfrak{M} = \mathcal{H}_\lambda$, and by normality of $\mathfrak{M}$, $\mathfrak{M}$ is isomorphic to $\mathcal{H}_\lambda$ and contradicts the definition of $\mathcal{H}$.

In the proof of Lemma 4.45 we will actually only need what has already been proven, that $\mathfrak{B}$ is a $\sigma$-cumulative. $\sigma$-cumulative sequences. The fact that $\mathfrak{B}$ is a $\mathcal{H}$-ultrafilter sequence will fail not during the proof.
of Lemma 6.4.

Proof of Lemma 6.4: We have to show that if $x \in p$, and $x \in \mathcal{F}$, then any $\mathcal{F}$ subset $x$ of $x$ is in a $\mathcal{F}(x, y)$-cone. Since $P$ is a $\mathcal{F}$-cone, we can take an ultrafilter sequence $\mathcal{F}$ in $\mathcal{F}$ so that $\mathcal{F}$ has a transitive system of indiscernibles. Then $\mathcal{F}$ is in $\mathcal{F}$ elementary, and $x$ in $\mathcal{F}$ is in $\mathcal{F}$ and $x$ is still a $\mathcal{F}$ cone, so $x$ is in a $\mathcal{F}(x, y)$-cone by Proposition 4.60.

Reduction of $\mathcal{F}$ cones: Most of the rest of Section 7 will follow up by the proof of Lemma 6.4 below. This lemma, which is the hardest part of the proof of Lemma 6.4, shows that $\mathcal{F}_{\alpha}$ is reducible to $\mathcal{F}_{\beta}$; the least ordinal $\beta$ such that there is a new $\mathcal{F}$ subset $x$ of $x$ in $\mathcal{F}_{\beta}$. We know that $\mathcal{F}$ is a $\mathcal{F}$-cone, and $\mathcal{F}$ is an ultrafilter sequence for $\mathcal{F}$, and that either $\mathcal{F}(x, y)$ - $\mathcal{F}$ or else $\mathcal{F}(x, y)$ is a limit ordinal. In addition we can assume that $\mathcal{F}(x, y)$ - $\mathcal{F}$ since otherwise the reduction is trivial. We also know that $\mathcal{F}$ is a $\mathcal{F}$-ultrafilter sequence but our proof of Lemma 6.4 will not use this fact or the basic assumption on $\mathcal{F}$. We restrict ourselves instead to the facts outlined above, so that we will be able to use Lemma 6.4 again in Section 8 in a slightly different context.

Following the proof of Lemma 6.4 we will show that the system of indiscernibles given by the reduction is in each $\mathcal{F}$ cone but complete. This last part, of course, will require an extra assumption on $\mathcal{F}$, and will also incidentally show that $\mathcal{F}$ is a $\mathcal{F}$-ultrafilter sequence above $\mathcal{F}_{\alpha+1}$. The fact that $\mathcal{F}$ is a $\mathcal{F}$-ultrafilter sequence at $\mathcal{F}_{\alpha+1}$ will hold once we know that the $\mathcal{F}_{\alpha+1}$ cone
\( \mathcal{L}_1 \) refers, more exactly, to any \( \mathcal{L}_1 \) definable sequence for \( \mathcal{L}_1 \), and hence a \( \mathcal{L}_1 \) definable measure for \( \mathcal{L}_1 \).

Following the proof that \( \mathcal{L}_1 \) has a basis, we complete a set of

indiscernibles, it will only be necessary to verify that the reduction yields a \( \Delta_1 \) rule.

A 3. Lemma: If \( p \in \mathcal{L}_1 \) is a monadic in \( \mathcal{L}_2 \) and \( p \) is the least parameter

such that \( \mathcal{L}_2 \) has a new \( \mathcal{L}_1 \) subset \( x \) of \( p \) with parameter \( x \) then \( \mathcal{L}_2 \) is reducible to \( p \) via \( p \).

The case we are interested in \( p \in \mathcal{L}_1 \) by 3.1. In 3.1, we state the lemma more

generally for the purpose of invariance: we assume \( p \) as an \( \mathcal{L}_1 \) definable

sequence that the lemma is true for \( p \in \mathcal{L}_1 \); otherwise, if \( p \in \mathcal{L}_1 \) and \( \exists x \mathcal{L}_1 \) definability of indiscernibility and existence are made inside \( \mathcal{L}_2 \). We will use

the hypothesis in the following form.

A 4. Proposition: If \( p \in \mathcal{L}_1 \) then the \( \mathcal{L}_1 \) hypothesis [1]

implies: (i) \( p \in \mathcal{L}_1 \) is the least parameter adding a new \( \mathcal{L}_1 \) subset of \( \mathcal{L}_1 \) and

(ii) \( p \in \mathcal{L}_1 \) is universal: any \( \mathcal{L}_1 \) subset of \( (\mathcal{L}_1) \) is \( \mathcal{L}_1 \) with

parameters from \( p \in \mathcal{L}_1 \).

Proof: Suppose that \( \mathcal{L}_1 \) is the least parameter adding a new \( \mathcal{L}_1 \) subset of

\( \mathcal{L}_1 \). Then \( \mathcal{L}_1 \)) \( \mathcal{L}_1 \) is \( \mathcal{L}_1 \) with \( \mathcal{L}_1 \) subset of \( \mathcal{L}_1 \). If \( u =

\mathcal{L}_1 \) then \( \mathcal{L}_1 \) \( \mathcal{L}_1 \) is such a set. By

the induction hypothesis \( \mathcal{L}_1 \) is reducible to \( \mathcal{L}_1 \) via \( \mathcal{L}_1 \). Thus have

the universality property. In particular \( \mathcal{L}_1 \) \( \mathcal{L}_1 \) \( \mathcal{L}_1 \) \( \mathcal{L}_1 \) for any

\( \mathcal{L}_1 \).
Since the proof of lemma 6 is rather long, we will first give an outline. Let us start by recalling the relevant concepts in the line structure of $L$. In this case $N_4$ is a canonical $L_4$ code for some $L_4$ and $n$ is a $L_4$ subset of $T \subset N_4$. We want to show that the $L_4$ ball of $p \mid \mathbb{P}$ in $N_4$ is all of $N_4$. Since $p \mid \mathbb{P}$ in $N_4$ is all of $N_4$, then $B$ is also a canonical $L_4$ code for some $L_4$. The converse is $L_4$, but $n \in L_4$, since it is in $N_4$. Similarly, $n \not\in L_4$. We conclude $n \neq a$, and hence $p \not\in L_4$ since they are both canonical $L_4$ codes for $L_4$. But $p \not\in \mathbb{P}$ and since $p$ is the least parameter defining $n$ we must have $c(p) = p$. But then $c(p)$ is the identity and as $L_4$ the $L_4$ ball of $p \not\in L_4$ we get required.

Our proof will start in the same way as letting $n \not\in L_4$ being all of $N_4$. We have $N_4$ and $D$ are both $L_4$ codes, but for structures $N_4$ and $D$ with different sequences of sequences, then we used to use invented criteria to match the sequence $N_4$ and $D$, while preserving the $L_4$ code $N_4$ and $D$ after introducing some machinery for dealing with such iterated linear codes.
we will prove

A.12 Lemma. Suppose \( B_1 = \langle \beta_1, \gamma_1, \delta_1 \rangle \) and \( B_2 = \langle \beta_2, \gamma_2, \delta_2 \rangle \) are \( \Sigma_k \) codes for some \( k \). Then there are isomorphisms \( 1 : B_1 \to B_2 \) and \( 1 : B_2 \to B_2 \) so that \( B_1 \) and \( B_2 \) are \( \Sigma_k \) codes for \( \beta_1 \) and \( \beta_2 \), respectively.

The next step is to show that \( B_1 = B_2 \). This is easy, provided that \( \pi : \beta_1 \to \beta_2 \) is id on an that \( \gamma_1 \), is still \( \Sigma_k \) admissible to \( \beta_1 \) and \( \beta_2 \). Otherwise let \( \beta \in \beta \) be least such that \( (\beta) : \beta \to \beta_0 \) is \( \Sigma_k \) admissible to \( \beta_1 \) and \( \beta_2 \). Then \( \beta \) is also the least ordinal such that the sequence \( \gamma_1 \) in \( \Sigma_k \) differs at \( \beta \) from the sequence \( \gamma_2 \) in \( \Sigma_k \), so we must have \( \beta \in \beta \). If \( \beta = \beta_0 \) then \( \beta \) is \( \beta \) and \( \beta \). We still show by a similar admissibility argument that in this case there is a \( \Sigma_k \) code for \( B_1 \), which will be used in the argument of \( \beta \) to show \( B_1 = B_2 \).

We define (1) lifts to diagrams
For a fixed and at the top level we can show that (i) forms a triangle:

Now we show that diagram (ii) preserves the canonical parameters:

Since the order is preserved,

and (ii) also forms a triangle. Thus we show by induction on \( n \) that (i) forms a triangle and the parameters are preserved for all \( n \). Since \( \delta \) is the identity and \( \Delta = \Psi(\delta(p)) \), it follows that diagram (i) commutes:

At this point it is possible, by an induction on \( n \), to show that all of the triangles (ii) commute. We only need to show that \( \delta \) is in the identity, and hence the indistinguishable necessary for the induction.

The rest of this argument (and hence of the paper) is in the proof that the parameters are preserved, i.e., that \( \delta(p) = \delta(p) \) for each \( p \).
We will have more to say about this way when the time comes to prove (1).

Supposing in general, suppose that $B = (N, D, S, s)$ is a $D^N$ code for some $D$ and that $f_{B^N}: B \rightarrow B_N$ is an iterated ultrapower which may lack an ultrapower in the sense of $S_{B^N}$ where $S_{B^N} = \text{end}(B_N)$, or $S_{B}$ an ultrapower of $S_{B}$. Then $f_{B^N}$ satisfies the condition for $D^N$ formulae. We want to show $B_{B^N}$ is a $D^N$ code for a structure $\mathcal{M}$ and that

Let $f_{B^N}$ extend a map $f^*: \mathcal{M} \rightarrow \mathcal{M}$. The map $f^*$ will be called an ultrapower of depth $k$. A single iterated ultrapower is said to be of depth $0$. Here precisely:

**Definition:** (i) If $B$ is a $D^N$ code then an iterated ultrapower of $B$ of depth $k$ is an iterated ultrapower, in general from that is, if $B^{\uparrow k} = \text{end}(B^{k})$, and $B^{k}$ is a map of the form

$$B^{k} = \{ a \in B^{k} | \forall a \in B^{k} : a = f^*(a) \}$$

where $B^{k_1}$ and $B^{k_2}$ have $B^{k_1}$ codes $B^{k_2}$ and $B^{k_2}$ respectively, and $f^*$ is the actual extension of an iterated ultrapower

$$B^{k_1} \rightarrow B^{k_2}$$

of depth $k_1$. Each $f^*$ is an iterated ultrapower of depth $k$ such that $f^*(\text{end}(B^{k_2}))$ is the identity.

The qualified term "iterated ultrapower" will always mean an iterated
algorithms of depth 0.

Proposition: If \( \delta^m \) is a \( Z_2 \times Z_2 \) code for \( \delta \) and \( \delta^m \rightarrow \delta^m \) is an idempotent ultrametry then for each \( k \leq m \). \( \delta^k \) is a \( Z_2 \times Z_2 \) code for a structure \( \delta^k \) and \( \delta^k \rightarrow \delta^k \) extends to an idempotent ultrametry \( \delta^k, \delta^k \rightarrow \delta^k \) of depth \( m-k \) between the \( Z_2 \times Z_2 \) codes.

In particular, \( \delta^k \rightarrow \delta^k \) extends to a map

\[
\delta^k, \delta^k \rightarrow \delta^k.
\]

We are now ready to prove Lemma 6.5. In using the lemma we will use details of the construction of the maps \( \delta^k \) and \( \delta^k \rightarrow \delta^k \) as well as their extensions. Also, although we have only the cases of equality stated it may for two codes \( \delta^k \) and \( \delta^k \) we will also apply it (to Lemma 6.5) with reasonable ease.

Proof of 6.5: For the sake of the induction we prove a slightly more general result: suppose that \( \delta_1 \) and \( \delta_2 \) are \( Z_2 \times Z_2 \) and \( Z_2 \times Z_2 \) codes, respectively, and suppose that the lemma is true for \( Z_2 \times Z_2 \) and \( Z_2 \times Z_2 \) codes whenever \( \delta_1 \times \delta_2 \times \delta_1 \times \delta_2 \times \delta_1 \times \delta_2 \). It is straightforward to define iterated ultrametries \( \delta_1, \delta_2 \rightarrow \delta_1 \times \delta_2 \times \delta_1 \times \delta_2 \times \delta_1 \times \delta_2 \) such that if \( p \) is the minimal of \( \delta_1 \) and \( \delta_2 \) we have \( \delta_1, \delta_1 \rightarrow \delta_1, \delta_2 \rightarrow \delta_2, \delta_2 \rightarrow \delta_2 \), and for \( \delta_1 \times \delta_2 \), \( \delta_2 \rightarrow \delta_2 \), \( \delta_1 \times \delta_2 \rightarrow \delta_1 \times \delta_2 \), \( \delta_1 \times \delta_2 \rightarrow \delta_1 \times \delta_2 \), and so on.
where $x_i$ should be taken to mean the set $S_i$ defined over $R_i$ if $x = x_i$ if $\sigma(y) = \sigma(x_i)$, and similarly for $x_j$.

Now assume $p = \sigma(y_1)$ (i.e., $p$ is the $\sigma$-coding of $x_1$). By Proposition 4.41 $R_i$ and $R_j$ are $S_{i-1}$ and $S_{j-1}$ codes. Let $R_k$ be the $\sigma$-coding of $x_2$. Then by the induction hypothesis $R_i$ and $R_j$ have iterated ultrapowers $^{R_i}R_i$ and $^{R_j}R_j$ where $^{R_i}R_i$ and $^{R_j}R_j$ are $S_{i-1}$ and $S_{j-1}$ codes of structures $^R_i$ and $^R_j$, as required. Since $^{R_k}R_k 
eq R_k$, these iterated ultrapowers will not move $p$. Enter $x_1$ in $S_{i-1}$ code for $^R_i$, and $x_2$ in $S_{j-1}$ code for $^R_j$, and $x_1 x_2$ in $S_{i-1}$ code for $^R_i$ and $S_{j-1}$ code for $^R_j$, where $\sigma(x_1) = x_j$ and $\sigma(x_2) = x_i$, we would use codings of both $^{R_i}R_i$ and $^{R_j}R_j$. 

We are now ready to return to the proof of Lemma 6.46. Recall that $x$ is a new $S_{i-1}$ subset of $x_1$ in $R_i$, with parameter $p$, and we have defined $y$ and $z$ by

\[ x = \sigma(x_1) = \sigma(z) \]

Then Lemma 6.46 gives us iterated ultrapowers $^R_i R_i$ and $^R_j R_j$.

\[ x \longrightarrow x_i \]

\[ x \longrightarrow x_j \]

where $x_i$ and $x_j$ are $S_{i-1}$ codes for structures $^R_i$ and $^R_j$, respectively. We need to show first that $x_i = x_j$, so that the diagram becomes
Let $S$ and $S'$ be each $\mathcal{E}$-codes for the same structure $\mathcal{A}$.

Proof. Let $\mathcal{B}$ be the least ordinal such that $|\mathcal{B}| > \alpha$ or $|\mathcal{B}| > \beta$, and suppose that there is a set $\mathcal{B}$ with parameters from $\mathcal{B}^\cup$. Then $\mathcal{B}$ is still $\mathcal{E}$-definable in $\mathcal{B}$, and so $\mathcal{B}$ is $\mathcal{E}$-definable in $\mathcal{B}$, but not a member of $\mathcal{B}$ itself. Then $\mathcal{B}$ is definable in $\mathcal{B}$, but not a member of $\mathcal{B}$ itself. Thus we must have $\mathcal{B}_1 = \mathcal{B}_2$.

Then $\mathcal{B}$ will be enough to show that there is a set $\mathcal{B}$ with parameters from $\mathcal{B}^\cup$. If $\mathcal{B} \not\subseteq \mathcal{B}$, this is immediate, since $\mathcal{B} \subseteq \mathcal{B}$.

Furthermore, $\mathcal{B}$ is the least ordinal such that $\mathcal{B}$ is in $\mathcal{B}_2$, and in $\mathcal{B} = \mathcal{B}^\cup$, $\mathcal{B}$ is definable as we must have $\mathcal{B}^\cup = \mathcal{B}^\cup$. Then we can assume $\mathcal{B} \subseteq \mathcal{B}^\cup$ if $\mathcal{B}$ is in $\mathcal{B}_2$, and $\mathcal{B}$ is in $\mathcal{B}^\cup$. We must have $\mathcal{B} = \mathcal{B}^\cup$ if $\mathcal{B}$ is in $\mathcal{B}_2$, since $\mathcal{B}^\cup$ would be in $\mathcal{B}^\cup$ as we must have $\mathcal{B}^\cup$.

Thus, if $\mathcal{B} = \mathcal{B}^\cup$, $\mathcal{B} = \mathcal{B}^\cup$ if $\mathcal{B} = \mathcal{B}^\cup$. Then $\mathcal{B} = \mathcal{B}^\cup$. The new subset $\mathcal{B}$ of $\mathcal{B}$ is $\mathcal{E}$-definable in $\mathcal{B}$ from parameters $\mathcal{B}^\cup = \mathcal{B}^\cup$.

Get $T$ (Fig. 1) and consider the iterated algebras.
We claim that for some $a \in V$, $\lambda_a$ is $\square$-definable in $\mathbb{S}^a$. Suppose not. $\lambda_a$ is certainly $\square$-definable in $\mathbb{S}^a$ by the definition of ultrapowers. Now apply Lemma 6.8 simultaneously to all of the structures $\mathbb{S}^a$, obtaining an ordered ultrapower $\lambda_a : \mathbb{S}^a \rightarrow \mathbb{S}^a$ such that $\mathbb{S}^a$ is a $\square$-code for $\square$. We know that $\lambda_a : \mathbb{S}^a \rightarrow \mathbb{S}^a$ is $\square$-definable in $\mathbb{S}^a$ and hence $\lambda_a \vdash \Box \phi$, providing a definite definable chain of ordinals.

This contradicts showing that $\lambda_a$ is $\square$-definable in some $\mathbb{S}^a$, and we will complete the proof of theorem 6.10 by showing that this implies that there is a new $\square$-definable $a'$. Suppose that $\lambda_a$ is defined in the $\square$-definable formula

$$\square \phi : \mathbb{S}^a \rightarrow \mathbb{S}^a$$
In either case \( a' \) is a \( \mathbb{R}^2 \) subset of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Since \( a \neq b \)'s, \( b \) is a \( \mathbb{R}^2 \) function with parameters in \( a \cup b \). In order with parameters in \( a \cup b \). Also the map \( f \rightarrow f \) is in \( b \), since \( f(\mathbb{R}^2) \). 

\( \text{(in L,5113,5131)} \): It follows that \( a' \neq b \), since otherwise \( a' \) would be an \( \mathbb{R}^2 \), so \( a' \) is the required new subset of \( a \). 

We now have the triangle of maps:

\[
\begin{array}{ccc}
\text{a} & \xrightarrow{f} & \text{b} \\
\downarrow & & \downarrow \\
\text{a'} & \xrightarrow{g} & \text{b'}
\end{array}
\]

Since all the maps preserve cases and the \( \mathbb{R}^2 \) case of \( f \) is unique this gives a triangle of maps of the \( \mathbb{R}^2 \) cases:

\[
\begin{array}{ccc}
\text{a} & \xrightarrow{f} & \text{b} \\
\downarrow & & \downarrow \\
\text{a'} & \xrightarrow{g} & \text{b'}
\end{array}
\]

Now we are ready to work our way down. Recall that for each \( m \) we have the maps:

\[
\begin{array}{ccc}
\text{a} & \xrightarrow{f_m} & \text{b} \\
\downarrow & & \downarrow \\
\text{a'} & \xrightarrow{g_m} & \text{b'}
\end{array}
\]

between the \( \mathbb{R}^2 \) cases.
Lemma. For each \( a, b \in R^m \), the diagram 

\[
\begin{array}{c}
\text{Input} \\
\downarrow a \\
\text{Output} \\
\downarrow b
\end{array}
\]

commutes.

The proof of (4.11) depends on a preliminary result. Recall that \( R_0 \) is the canonical partition used to define \( R_{m-1} \).

Lemma: [preservation of parameters] Suppose \( a, b \) and \( R_0 \) such that if \( m = n \) then 

\[
\lambda_0^m(v^{(a)}) = \lambda_0^n(v^{(b)}),
\]

and if \( m < n \) then 

\[
\lambda_1^m(v^{(a)}) = \lambda_1^n(v^{(b)}).
\]

Proof of Lemma 4.11: We first prove by induction on \( k \), \( g, h \) and \( t \) that \( R_0 \) is preserved. For each \( R_0 \), we can suppose that \( m \neq n \) and \( t = n \), or \( t = m \). By Lemma 4.10 we have 

\[
\lambda_0^m(v^{(a)}) = \lambda_0^n(v^{(b)}), \quad \lambda_1^m(v^{(a)}) = \lambda_1^n(v^{(b)}).
\]

Not \( R_0 \) and \( R_0 \) are obtained by reducing \( R_0 \) and \( R_0 \), respectively, so we must have 

\[
R_0 = R_0.
\]

In particular we have \( R_0 \) such that \( R_0 \) is obtained by reducing \( R_0 \) and \( R_0 \), respectively, so we must have 

\[
R_0 = R_0.
\]

Hence, \( R_0 \) is the identity on \( \lambda_0^m(v^{(a)}) \) and \( \lambda_1^m(v^{(a)}) \) for all \( m, n \). Let \( S \in R_0 \) be defined from members of \( S \) by \( \lambda_0^m(v^{(a)}) = \lambda_1^m(v^{(a)}) \). Then \( S \) is the identity on \( \lambda_0^m(v^{(a)}) \), and \( S \) is the identity on \( \lambda_1^m(v^{(a)}) \) for all \( m, n \). Hence, \( S \) is the identity on \( \lambda_0^m(v^{(a)}) \), and \( S \) is the identity on \( \lambda_1^m(v^{(a)}) \) for all \( m, n \). Hence, the lemma holds.

\( \square \) (4.11) Proof (4.11)

The proof of (4.11) will now complete the proof of Lemma 4.11, except for showing that \( \lambda_0^m(v^{(a)}) = \lambda_0^n(v^{(a)}) \) for all \( m, n \). Hence, the lemma is essentially the same for the cases \( m < n \) and \( m > n \), as for uniformly we set.
and we must show that \( p_{\text{l}}(s) \leq p_{\text{r}}(s) \). Let us briefly outline the proof. The direction that \( p_{\text{l}}(s) \leq p_{\text{r}}(s) \) is a reasonably straightforward extension of Jeavons's argument. We note that \( p_{\text{l}}(s) \) is a new \( \Omega \) subset of \( p_{\text{r}}(s) \) with parameter \( p_{\text{r}}(s) \) and that \( p_{\text{l}}(s) \) preserves this fact. From this, we prove a lemma which shows the inductive hypothesis that \( p_{\text{l}}(s) \) (in fact any \( \Omega \) subset of \( p_{\text{r}}(s) \)) preserves the fact that \( p_{\text{r}}(s) \) is minimal for adding any new \( \Omega \) subset. These two facts imply that \( p_{\text{l}}(s) \leq p_{\text{r}}(s) \).

The other direction is more straightforward. It doesn't arise in Jeavons's proof. If \( a = b \) then \( x_{a} \) is the identity and \( x_{b} \) is a simple collapse on \( [p_{\text{r}}(s)]_{a} \). In immediate, whereas for \( a \neq b \) lemma 4.1 follows exactly from the fact that the embeddings, being extensions of embeddings at lower levels, are all strong enough to preserve the inductive assertion that \( x_{a} \) is minimal. We are unable to simply reverse the first part of the proof because we have been unable to prove directly that \( p_{\text{r}}(s) \) is the least parameter.
adding a new \( \mathcal{S} \) subset of \( \mathcal{R} \). What we do instead is to first use the
inductive hypothesis again to prove that \( A_{\mathcal{R}}(n) \) is universal, adding every
new \( \mathcal{S} \) subset of \( \mathcal{R} \). Thus if \( A_{\mathcal{R}}(n) \leq A_{\mathcal{R}}(n+1) \) we have that
\( A_{\mathcal{R}} \) is an ordered ultrafilter in which \( A_{\mathcal{R}}^{\mathcal{S}}(q) \) is set minimal.
For adding the specific new \( \mathcal{S} \) subset \( A_{\mathcal{R}}^{\mathcal{S}}(q) \), we can lift this to \( \mathcal{R} \),
letting \( \delta : \mathcal{R} \to \mathcal{S} \) be the desired ultrafilter whose support is the image
of \( \delta \), so we will show that \( \delta[q] \) is set minimal. For adding the new \( \mathcal{S} \)
subset \( \delta[q] \) of \( \mathcal{S} \). This contradicts our lemma from the first direction,
which asserted that \( \delta[q] \) is maximal for any new \( \mathcal{S} \) subset in any
ultrafilter - note that our originally presumable result has become critical here.

We are not quite done: we have to see that the induction hypothesis is
in fact satisfied. For \( n \geq 0 \), there is no problem: the induction hypothesis
was the assertion that \( \mathcal{Q}_n \) is a \( \mathcal{S}_n \)-code. For \( n > 0 \), the induction
hypothesis is less strict: we were able to use the minimality of \( \mathcal{Q}_n \) to show
that \( \mathcal{Q}_n \) is a \( \mathcal{S}_n \)-code. Now for the first direction the induction hypothesis was the
assertion that \( \mathcal{Q}_n \) is a \( \mathcal{S}_n \)-code for some \( n \). This is true by the choice of \( \mathcal{Q}_n \) as we have \( \mathcal{Q}_n \leq \mathcal{Q}_{n+1} \).
The other direction was the minimality of \( \mathcal{Q}_n \), which is almost exactly what we are
trying to prove: we still need a different argument to show that \( \mathcal{Q}_n \) is
\( \mathcal{S}_n \)-code. We can give the argument here in detail: since \( \mathcal{Q}_n \) is a\nultrafilter, \( \mathcal{Q}_n \leq \mathcal{Q}_{n+1} \), and it is enough to show that \( \mathcal{Q}_n \) is a \( \mathcal{S}_n \)-code. Let
\( \delta \) be the first minimal element for \( \mathcal{Q}_n \). Then \( \delta \) is \( \mathcal{S}_n \)-code, and we would have
\( \delta \leq \mathcal{Q}_{n+1} \). In fact we must have \( \delta \leq \mathcal{Q}_{n+1} \) or we would have
\( \delta \leq \mathcal{Q}_n \). This would
Lemma 10.1: If $\Delta$ is an algebraic variety and $\phi: \Delta \to \mathbb{P}^1$ is a morphism, then $\phi$ is an isomorphism if and only if $\Delta$ is a connected algebraic variety.

Proof: Let $\phi: \Delta \to \mathbb{P}^1$ be a morphism. Then $\phi$ is an isomorphism if and only if $\phi$ is a birational map and $\Delta$ is connected. If $\phi$ is a birational map, then it is surjective and $\Delta$ is connected. Conversely, if $\phi$ is a birational map and $\Delta$ is connected, then $\phi$ is an isomorphism.

We now prove the detailed proof of Lemma 10.1. If $\phi: \Delta \to \mathbb{P}^1$ is a birational map and $\Delta$ is connected, then $\phi$ is an isomorphism. We begin with a general result concerning the structure of connected algebraic varieties.

Corollary 10.2: Suppose that $\alpha$ is a bounded $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ which is not $\mathbb{P}^2$-divisible. Then $\alpha$ has a bounded $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ which is not $\mathbb{P}^2$-divisible.

Proof: The bounded $\mathbb{P}^1$-bundle $\alpha$ can be factored as
$$
\alpha: \mathbb{P}^2 \to \mathbb{P}^1 \to \mathbb{P}^2 \to \mathbb{P}^2 \to \mathbb{P}^2
$$
where $\alpha_1$ is of depth $1$, $\alpha_2$ is of depth exactly $1$, and $\alpha_3$ is relatively of depth greater than $1$. Notice that the conclusion of the proposition is a bounded $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ statement.

We wish to prove that $\phi$ is an isomorphism. This is why we have to factor $\phi$ into three parts. The main problem arises in $\phi_1$: we will show that $\phi_1$ preserves the property of not being a section of $\phi_2$ for every $x$. Since the properties of not being a section of $\phi_2$ for every $x$, we conclude that $\phi_3$ is an isomorphism and $\phi_1(\Delta)$ is not a section of $\phi_2$.
We consider \( S_1 \). First, suppose that \( S_2 \neq S_1 \) and \( x_1 \in S_2 \). Then, any \( S_2 = \{ x \} \) is the limited ultrapower \( S_1 \), we can assume \( x \in \text{supp}(f) \) is not in the support of \( f \). Since \( \text{supp}(f) \) is not an extended ultrapower, there must exist a bound \( x \) such that \( x \neq f(x) \). But then we have \( x \in S_1 \), \( x \not \in S_2 \), contradicting the maximality.

Now we come into the \( \mathcal{M} \) code \( S_2 \) of \( S_1 \) the assertion that \( \mathcal{M} \subseteq \mathcal{N} \).

where \( S_1 \) is defined by the \( \mathcal{M} \) formula \( t \). This asserts that there is \( y \) such that:

\[
\forall x \in S_1 \exists y \in S_1 \exists z \in S_1 \phi(x, y, z) \quad (6)
\]

where \( \phi \) is the formula in the \( \mathcal{M} \) code \( S_2 \). The assertion that \( S_2 \) is an \( \mathcal{M} \) code:

\[
\forall x \in S_1 \exists y \in S_1 \phi(x, y, z) \quad (6)
\]

where \( \phi \) is the \( \mathcal{M} \) formula (with parameter \( b(x) \)) defining the complement of \( S_1 \). Thus the assertion that \( S_2 \) is not an \( \mathcal{M} \) code of \( S_2 \), where \( S_2 \) is the \( \mathcal{M} \) and higher order codes of \( S_2 \). Thus both assertions are preserved by \( S_2 \).

Now suppose that \( S_2 \neq S_1 \) but \( S_1 \not \subseteq S_2 \); note that:

\[
\forall x \in S_2 \exists y \in S_2 \exists z \in S_2 \phi(x, y, z) \quad (6)
\]

where \( \phi \) is the formula in the \( \mathcal{M} \) code \( S_2 \). Since \( S_2 \neq S_2 \), there is a \( S_2 \)-function \( f : S_2 \rightarrow S_2 \) in \( S_2 \) such that for all \( x \in \text{dom}(f) \) we have \( \forall y \in S_2 \exists z \in S_2 \phi(x, y, z) \). We consider the support of \( f \) and \( x \)-ultrapower at a time. First, an ultrapower of a measure on \( \mathcal{M} \) cannot contain a measure since it is in \( \mathcal{M} \). Furthermore, if the \( \mathcal{M} \)-ultrapower is bounded on a set of measure \( \text{supp}(f) \), then it is a
number of $B_1$, since its graph is rudimentary. Thus we can have trouble only at the cofinality $\beta$ of $B_1$, we can factor $s_1$ into
$s_1 = y_1 \theta B_0 \theta B_1 \theta B_2 \theta B_3 \theta B_4 \theta B_5 \theta B_6 \theta B_7 \theta B_8 \theta B_9$,
where $\alpha_0$ is the recorded antihomomorphism above $\beta$. $\alpha_1$ in either the identity
to a single antihomomorphism $\alpha_0$ is at $\beta$, and $\alpha_2$ is the recorded antihomomorphism
$s_1$ below $\beta$. Thus $\alpha_0$ and $\alpha_1$ both preserve the property either of not
being a number or (on the same argument) of not being $A_0$ and it is enough
to show that if $s_1 = s_1 \theta B_2 \theta B_3 \theta B_4 \theta B_5 \theta B_6 \theta B_7 \theta B_8 \theta B_9 \theta B_{10}$

So above, let $s_1 = s_1 \theta B_2 \theta B_3 \theta B_4 \theta B_5 \theta B_6 \theta B_7 \theta B_8 \theta B_9 \theta B_{10}$,
and define a function

\[ f : \mathbb{N} \rightarrow \mathbb{N} \]

so that

(1) \( f(n) = \alpha_0(\alpha_1(n)) \) whenever possible. We can take $\alpha_0$ to be $\mathbb{N}$ in $\mathbb{N}$, and since $\alpha_1$ is
countably complete we can take $\alpha_1$ so that if $f(n)$ is defined \( n \in \mathbb{N} \)

\[ Y = \{ n \in \mathbb{N} : f(n) = \alpha_1(n) \} \] \( \notin \mathbb{N} \).

Since both domain \( Y \) and $\mathbb{N}$ are subsets of $\mathbb{N}$, the $\mathbb{N}^+$
projection of $\mathbb{N}$ we can take \( \alpha_0 \) to be a total function with
domain \( Y \subseteq \mathbb{N}^+ \). Furthermore $\alpha_0$ is undefined in $\mathbb{N}$. For all $n \in \mathbb{N}$ we have \( s_1 \theta B_2 \theta B_3 \theta B_4 \theta B_5 \theta B_6 \theta B_7 \theta B_8 \theta B_9 \theta B_{10} \theta B_{11} \theta B_{12} \theta B_{13} \theta B_{14}$

and the definition of $\alpha_0$ we have

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\[ Y = \{ n \in \mathbb{N} : f(n) = \alpha_1(n) \} \] \( \notin \mathbb{N} \).
giving a \( \mathcal{D} \) definition of the complement of \( \mathcal{A} \).

If \( \mathcal{A} \) is not defined as a set in \( \mathcal{D} \) then \( \mathcal{A} = \mathcal{D} \) iff

where \( \mathcal{A} = \text{def} \{ x \mid \text{hi} \} \) gives the \( \mathcal{D} \) definition. Hence \( \mathcal{A} \) is \( \mathcal{D} \) as was to be shown.

\[ \square \]

Notice that this does not give the stronger result that there is no set \( \mathcal{A} \in \mathcal{D} \) such that \( \mathcal{A} \in \mathcal{D} \) iff \( \mathcal{A} \in \mathcal{D} \) for all \( \mathcal{A} \in \mathcal{D} \). In fact if the ordinality of the projection \( p \) of \( \mathcal{D} \) is greater than the ordinality of \( \mathcal{D} \) we do not have an example of such a case for it seems that there is no \( \mathcal{A} \in \mathcal{D} \) such that \( \mathcal{A} \in \mathcal{D} \). However the proof does imply that if \( \mathcal{D} \in \mathcal{D} \) then \( \mathcal{D} \in \mathcal{D} \), where \( \mathcal{D} \subseteq \mathcal{D} \) is the projection of \( \mathcal{D} \).

A \( \mathcal{D} \)-variant of \( \mathcal{D} \) is a new \( \mathcal{D} \) subset of \( \mathcal{D} \) with parameter \( \mathcal{D} \).

**Proof:** If \( m \leq n \) then \( \mathcal{A} \) is a universal \( \mathcal{D} \) set in \( \mathcal{D} \) and hence \( \mathcal{A} \in \mathcal{D} \) is a universal \( \mathcal{D} \) set in \( \mathcal{D} \). Hence \( \mathcal{A} \in \mathcal{D} \) set by the lemma \( \mathcal{D} \in \mathcal{D} \) is set in \( \mathcal{D} \).

If \( m > n \) then \( \mathcal{A} \) is not set known to be universal, but \( \mathcal{A} \) in the identity set is of depth \( n \) and hence preserves the fact that \( \mathcal{A} \in \mathcal{D} \).

\[ \square \]

A \( \mathcal{D} \)-variant \( \mathcal{D} \) is in any desired ultravariant (possibly of depth greater than \( n \)). Then \( \mathcal{A} \in \mathcal{D} \) is the local parameter adding a new \( \mathcal{D} \) subset of \( \mathcal{D} \) in \( \mathcal{D} \).
where $n_x$ is of depth $x$ and $s_x$ is arbitrary of depth greater than $x$.

First we look at the easier half. $s_1$; we suppose that $s_1 = 0 < n_0(1) = 0 = n_0(1)$ and show that $s_1 \notin D_2$. In $D_2$, define $y = 0 < n_0(1) = 0 = n_0(1)$. Then $y \notin D_2$ and it is enough to show that $s_1 = 0 < n_0(1)$.

To this purpose we define $y \in x$: for $x \in x$, $y \in x$ then $y$ is arbitrary of depth greater than 0.

The case $n_x$ is slightly more complicated. Suppose $n_x = 0 < n_0(1) = 0 = n_0(1)$. Then by the last paragraph $s_1 = 0 < n_0(1) = 0 = n_0(1)$. In other words, $s_1 = 0 < n_0(1)$. Hence $y \in x$.

For $x \in x$, if $x = 0 < n_0(1)$ we have $x = 0 < n_0(1)$.

We have $s_1 = 0 < n_0(1)$ and $s_1 = 0 < n_0(1)$.

Then $n_x$ is arbitrary of depth greater than 0.

The relation is transitive, satisfying $n_x = n_y$, and is well-founded in the sense that there is no infinite descending chain. $s_1$ has the same properties as the measurable complement of the stratification, so we can choose $s_1$ consistent with $s_2$ and $s_3$. Therefore, we can continue to choose $s_1 = 0 < n_0(1)$.
where \( \Pi \) is the image of the translated operator \( S_g \) by the map \( \eta_g \). Then the diagram can be completed by \( T \), which has been factored into maps \( \delta_{T} \) and \( \eta_{T} \), which is the identity on \( \eta_{T} \).

Diagram 1: \( S_{g+1} \), \( g' = [1] \) in \( \mathcal{M} \), and \( \delta_{gT}(g) = \eta_{T}(g) \). Then \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \). as \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \). We then see that and hence is not equal to \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \). That is, \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \).

This is still true in \( \mathcal{M} \) since \( \eta_{gT} = \eta_{T} \). We now need to worry about the case \( u < v < w < [0] \). That is, we have the triangle

\[
\begin{array}{ccc}
\eta_{gT} & \to & \eta_{gT} \\
\downarrow & & \downarrow \\
\delta_{gT}(g) & \to & \delta_{gT}(g)
\end{array}
\]

where \( \eta_{gT} \) is the image of the translated operator \( S_{g} \) by the map \( \eta_{g} \). Then the diagram can be completed by \( T \), which has been factored into maps \( \delta_{T} \) and \( \eta_{T} \), which is the identity on \( \eta_{T} \).

Diagram 1: \( S_{g+1} \), \( g' = [1] \) in \( \mathcal{M} \), and \( \delta_{gT}(g) = \eta_{T}(g) \). Then \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \). as \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \). We then see that and hence is not equal to \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \). That is, \( \delta_{gT}(g) = [1] \) in \( \mathcal{M} \).

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\[
\begin{array}{ccc}
\eta_{gT} & \to & \eta_{gT} \\
\downarrow & & \downarrow \\
\delta_{gT}(g) & \to & \delta_{gT}(g)
\end{array}
\]
This completes the proof of preservation of parameters, as we now know that the triangle:

\[
\begin{array}{c}
\text{l} \\
\text{b} \\
\text{c}
\end{array}
\]

commutes. We will see how that 1 is the identity. This will prove that

\( \mathcal{H}_1 \) is isomorphic to \( \mu \) via \( \mu \), using the system of indiscernibles generated by the iterated ultrapowers \( \mu \), and hence will complete the proof of Lemma 4.2.

First, we observe that \( 1[\mathcal{H}_1] \equiv \mu \). We have already used the fact that \( \delta^p - 1 : \mathcal{H}_1(\alpha) \) for some \( \lambda \in \mathcal{H}_1(\alpha) \) and hence \( \mu \equiv \mu \). But then

\[ \mu \equiv \mu \iff 1[\mathcal{H}_1] \equiv 1[\mathcal{H}_1] \equiv \mu \text{ as well.} \]

For the rest of the proof we use some of the notation from Theorem 2.5.

Since \( \Lambda \) is an ultrapower of \( \Lambda \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and \( \Lambda \rightarrow \mathbf{N} \) and 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First consider the case $\mathcal{P}(\omega) = \mathcal{P}_\omega$. Then $I_{\alpha}(\omega) = \omega$ and there is a $\mathcal{P}(\omega)$ function $f$ such that for some $\eta < \omega$, we have $\eta^\omega = \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$. Thus $\eta^\omega = \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ in $\mathcal{P}_\omega$. Since $\mathcal{P}(\omega, \mathcal{P}(\omega, \eta)) = \eta$, Lemma 4.40 implies that $\eta \in \mathcal{P}(\omega, \mathcal{P}(\omega, \eta)) \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$. But then there is $\xi < \omega$ such that $\xi \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ and hence $\xi \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta)) = \eta$. This is absurd as $\mathcal{P}(\omega, \mathcal{P}(\omega, \eta)) = \eta$ is impossible.

Now suppose $\mathcal{P}(\omega, \eta) < \eta$. Then there is $x \in \mathcal{P}(\omega, \eta)$ such that $x \notin \mathcal{P}(\omega, \eta)$ and $F_{\mathcal{P}(\omega, \eta)}(x, y)$. Again $\mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ for some $x < \omega$, so $\mathcal{P}(\omega, \eta) = \mathcal{P}(\omega, \mathcal{P}(\omega, \eta)) = \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$. Now the formula

$\exists x \in \mathcal{P}(\omega, \eta) \in \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ and $\mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ are elementary so there is $y \in \mathcal{P}(\omega, \eta)$ such that $y \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$. Then $\mathcal{P}(\omega, \eta) = \mathcal{P}(\omega, \mathcal{P}(\omega, \eta)) = \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$. Since $x \in \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$, we have $x \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ and hence $x \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$. Then $x \neq \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ and so $x \in \mathcal{P}(\omega, \mathcal{P}(\omega, \eta))$ contrary to the choice of $x$. Hence the case $\mathcal{P}(\omega, \eta) < \eta$ is impossible, as $\eta$ must be the identity.

We now know that $\mathcal{P}_\omega$ is reducible to $\mathcal{P}_{\omega+1}$, the $\mathcal{P}_{\omega+1}$ is. In order to complete the proof of Lemma 4.40 and hence of Lemma 4.1 we have to show that the reductions yields a $\mathcal{P}_\omega$-code of $\mathcal{P}_\omega$. There are two non-trivial things to be proved: that the reduction is rank $\omega$ complete, and that the structure $(\mathcal{P}_\omega, \mathcal{P}_{\omega+1})$ is weakly $\omega$-categorical.

**5.41 Lemma.** The reduction of $\mathcal{P}_\omega$ to $\mathcal{P}_{\omega+1}$ the $\mathcal{P}_{\omega+1}$ is rank $\omega$ complete.
Proof: We have to show that if $a = a_0 a_1 \cdots a_n b$ and $b \neq \perp$ then $\sigma(a, 0) \preceq_{(a, 0)} a$. We will show by induction over $n$ that $\sigma(a, 0) = a_0 a_1 \cdots a_n b$.

We first show that $\sigma(a, 0) \equiv_{a} a$. Since $\sigma(a, 0)$ is a $\Sigma_{a}$-sentence, this will ensure that $\sigma(a, 0)$ has all the similarity we will require.

Let $a$ be the least ordinal such that $\sigma(a, b) \neq \sigma(a, 0)$. Then $a$ belongs to $\sigma(a, 0)$ of $\sigma(a, 0)$ is the $\Sigma_{a}$-sentence $\sigma(a, 0) \equiv_{a} a$. But $a$ is not possible because $\sigma(a, 0)$ is in the $\Sigma_{a}$-sentence of $\sigma(a, 0) \equiv_{a} a$. So $a$ is $\sigma(a, 0)$ as well. Since $\sigma(a, 0)$ is $\Sigma_{a}$-sentence, it follows that $\sigma(a, 0)$ is also a sentence of $\sigma(a, 0)$. But $\sigma(a, 0)$ is defined as $\sigma(a, 0)$, so $\sigma(a, 0)$ is $\sigma(a, 0)$.

We now prove the lemma for the case $n = 0$: that is, we show that $\sigma(a, 0) \equiv_{a} a$ if $\sigma(a, 0) \equiv_{a} a$. We have $a = a_0$. Since $a = a_0$, we have $a = a_0$. In particular $a$ is a limit ordinal.

Now suppose $a = a_0 a_1 \cdots a_n b$. Recall that $\sigma(a, 0) = a_0 a_1 \cdots a_n b$. The proof is similar to the case $n = 0$.
We have to check that the set of \( x \) satisfying each of the clauses of definition (1) is in \( \text{Proj}.A \). By assumption \( \text{Proj}.A \times \text{Proj}.A \) is in \( \text{Proj}.A \) as clause (a) is immediate. Since any set described in an unbounded set of ordinals less than \( x \) must be unbounded in \( x \), clause (b) \((x) = (x)\) is bounded in \( x \) as \( x \in \text{Proj}.A \); clause (c) \((0) \in x \) is also in \( \text{Proj}.A \). By a normality argument: \( (x) \in (x) \Rightarrow (x) \subset x \Rightarrow x \subset (x) \). Thus the set of \( x \) satisfying clause (c) is in \( \text{Proj}.A \).

Clause (d) follows easily: we have \( x = \{ x \} \) and \( x = x \) for almost all \( x < x \). But by definition \( \text{Proj}.A \times \text{Proj}.A \) is bounded in \( x \), so every sufficiently large member of \( \text{Proj}.A \times \text{Proj}.A \) satisfying (a) also satisfies (c) and hence is in \( \text{Proj}.A \).

By only reasoning to define the \( \text{Proj}.A \times \text{Proj}.A \) case. We have a projection of \( \text{Proj}.A \) the \( \text{Proj}.A \) projection \( p_{\text{Proj}.A} \) of \( \text{Proj}.A \), the least \( d \) each of which takes a transitive collapse of the range of \( d \). Most of the conditions follow immediately, since \( p_{\text{Proj}.A} \) is a coding set and a coding function for the larger class of \( \text{Proj}.A \). We need to see that the range of \( b \) contains \( \text{Proj}.A \). For every element \( b \) of \( \text{Proj}.A \), the set of ordinals \( \eta_{\text{Proj}.A} \) for some \( \eta_{\text{Proj}.A} \) and sequence \( i \) since \( b \) is in
as defined in $R_{e}$, is a normal and comes a differential to eliminate the
sequences $s$.  

2.3.1.18