§5 Strong Sequences and Elementary Embeddings

In this section we return to the model $K(\mathcal{F})$, where $\mathcal{F}$ is a strong sequence. Our principal aim is to prove two basic theorems about $K(\mathcal{F})$:

<u>5.1 Theorem</u>: If $\mathcal{F}$ is strong then $K(\mathcal{F} \upharpoonright (\kappa,\lambda)) \subset K(\mathcal{F})$ for all pairs (κ,λ).

<u>5.2 Theorem</u>: If $\mathcal{F}$ is strong and $j:K(\mathcal{F}) \to Q$ is an iterated ultrapower then $j(\mathcal{F})$ is strong and $Q = K(j(\mathcal{F}))$.

These theorems will be corollaries of a more general result, Lemma 5.5. In order to state Lemma 5.5 in the generality which will be required in the next section we will first give some definitions.

<u>5.3 Definition</u>: (i) If $j:M \to N$ is an elementary embedding and $\delta \in N$ then N is j-generated from δ if $N = \{j(f)(x): f \in M \text{ and } x \in \delta^{<\omega}\}$.

(ii) If $J:M \to N$ is j-generated from δ, $M^* \supset M$, $\delta \leq j(\kappa)$, and $M^* \cap P(\kappa) \subset M$ then $j^*:M^* \to N^*$ is defined as follows: If $f \in M^*$ and $k \in \delta^{<\omega}$ then $[(f,x)] = \{(f',x'): f' \in M,\ x' \in \delta^{<\omega}$, and $(x,x') \in j(\{w,w'): f(w) = f'(w')\})$. We say $[(f,x)]E[(f',x')]$ if $(x,x') \in j(\{(w,w'): f(w) \in f'(w')\})$. Then N^* is the class of equivalence classes $[(f,x)]$. If N^* is well founded under E then we will identify it with its transitive collapse.

If $j:M \to M^\kappa/U$ then M^κ/U is j-generated from $\kappa+1$. More generally, if $j:M \to N$ is an iterated ultrapower then N is j generated from the least δ larger than all of the indiscernibles generated by j.

If $j:M \to N$ is an iterated ultrapower then $j^*:M^* \to N^*$ is simply the iterated ultrapower of M^* by the same ultrafilters as used for M. Notice that the condition $M^* \cap P(\kappa) \subset M$ implies that the ultrafilters in M are still ultrafilters in M^*. In this case, N^* will necessarily be well founded.

<u>5.4 Proposition</u>: If $\mathfrak{F}$ is strong, $\mathfrak{F}' \upharpoonright \ell(\mathfrak{F}) = \mathfrak{F}$, and $\mathfrak{F}'$ is an ultrafilter sequence above $\ell(\mathfrak{F})$ in $K(\mathfrak{F}')$ then $\mathfrak{F}'$ is strong and $P(\ell(\mathfrak{F})) \cap K(\mathfrak{F}') \subset K(\mathfrak{F})$.

<u>Proof</u>: That $P(\ell(\mathfrak{F})) \cap K(\mathfrak{F}') \subset K(\mathfrak{F})$ can be proved in the same way as 3.10(iv) was proved. It follows that $\mathfrak{F} = \mathfrak{F}' \upharpoonright \ell(\mathfrak{F})$ is an ultrafilter sequence in $K(\mathfrak{F}')$, and hence $\mathfrak{F}'$ is strong. □

<u>5.5 Lemma</u>: Suppose $\mathfrak{F}$ is strong, $\mathfrak{F}$ is a set, $j:K(\mathfrak{F}) \to Q$ is an elementary embedding such that Q is j generated from $j(\ell(\mathfrak{F}))$, and for all strong sequences $\mathfrak{F}'$ such that $\mathfrak{F}' \upharpoonright \ell(\mathfrak{F}) = \mathfrak{F}$, if $j^*:K(\mathfrak{F}') \to Q^*$ is the extension of j to $K(\mathfrak{F}')$ then Q^* is well founded. Then for all $\delta \leq j(\ell(\mathfrak{F}))$ every $j(\mathfrak{F}) \upharpoonright \delta$ mouse is in Q.

Lemma 5.5 is the promised general result. Before proving it we use it to prove Theorems 5.1 and 5.2.

<u>Proof of 5.1 and 5.2</u>: If $\mathfrak{F}$ is a set then Theorem 5.1 follows immediately from 5.5 by taking $j:K(\mathfrak{F}) \to K(\mathfrak{F})^{\kappa}/\mathfrak{F}(\kappa,\lambda)$ and $\delta = \kappa + 1$. To prove Theorem 5.2 we observe that $Q \models V = K(j(\mathfrak{F}))$, so $Q \subset K(j(\mathfrak{F}))$. By applying Lemma 5.5 with $\delta = j(\ell(\mathfrak{F}))$ we see that $K(j(\mathfrak{F})) \subset Q$, so $Q = K(j(\mathfrak{F}))$.

To apply Lemma 5.5 we have to verify that the hypothesis is satisfied. If $\mathfrak{F}'$ is a strong extension of $\mathfrak{F}$ then $j^*:K(\mathfrak{F}') \to Q^*$ is an iterated ultrapower. Since $K(\mathfrak{F}') \models$ ($\mathfrak{F}'$ is an ultrapower sequence) by Proposition 5.4, every iterated ultrapower of $K(\mathfrak{F}')$ lying in $K(\mathfrak{F}')$ is well founded. An absoluteness argument then shows that every iterated ultrapower of $K(\mathcal{F}')$ is well founded (see [Mitchell, 1974], Lemma 2.1), so Q^* is well founded as required.

If $\mathcal{F}$ is not a set then we can prove 5.1 and 5.2 by picking ν with $j(\nu) > \delta$ and applying Lemma 5.5 to the iterated ultrapower $j':K(\mathcal{F}\restriction\nu) \to Q'$ which is obtained by using the same ultrafilters on ordinals less than ν as for j, and skipping ultrapowers by ultrafilters on ordinals larger than ν. $\square$ 5.1, 5.2

<u>Proof of 5.5</u>: Suppose that $\mathcal{F}$, j, Q and δ are as given and that M is a $j(\mathcal{F})\restriction\delta$-mouse. We will show that $M \in Q$. We can assume that $\delta_M = \delta$; otherwise we could replace j by

$$j':K(\mathcal{F}) \xrightarrow{j} Q \xrightarrow{k} \mathrm{ult}_{\delta_M}(Q,\mathcal{F}(\kappa,0)) = Q'$$

where κ is the least ordinal greater than δ such that $c^{j(\mathcal{F})}(\kappa) > 0$. Then $j'(\mathcal{F})\restriction\delta_M = j'(\mathcal{F})\restriction\delta$, so $M \in Q$ if $M \in Q'$ because $k \in Q$. Also we can assume that Q is j generated from δ since otherwise we could replace j by $j':K(\mathcal{F}) \to Q'$, where Q' is the transitive collapse of $\{j(f)(\nu): \nu < \delta$ and $f \in K(\mathcal{F})\}$.

We will recursively define a sequence $\mathcal{F}'$ such that

(1) $\mathcal{F}'\restriction\ell(\mathcal{F})+1 = \mathcal{F}$,

(2) for all $(\kappa,\lambda) \in$ domain $(\mathcal{F}')$, if $\kappa > \ell(\mathcal{F})$ then $\mathcal{F}'(\kappa,\lambda)$ is a countably complete $K(\mathcal{F}'\restriction(\kappa,\lambda))$ ultrafilter, and

(3) For all $\kappa \geq \ell(\mathcal{F})$, $\mathcal{P}(\kappa)\cap K(\mathcal{F}') = \mathcal{P}(\kappa)\cap K(\mathcal{F}'\restriction\kappa+1)$.

Note that (2) implies that $\mathcal{F}'$ is strong above $\ell(\mathcal{F})$ by Theorem 3.11. It follows that if $\kappa \geq \ell(\mathcal{F})$ then $\mathcal{P}(\kappa)\cap K(\mathcal{F}')\subset K(\mathcal{F}'\restriction\kappa+1)$, so for (3) we only need to show that $\mathcal{P}(\kappa) \cap K(\mathcal{F}' \restriction \kappa + 1) \subset K(\mathcal{F}')$. A sufficient (and, in fact, necessary) condition for this to hold is that

(3′) If $\ell(\mathcal{F}) \leq \kappa' < \kappa \leq \ell(\mathcal{F}')$, N is a $\mathcal{F}\restriction \kappa' + 1$ mouse, and $\delta_N = \kappa' \cup \sup\{\delta_\gamma : \gamma \leq \kappa'\}$ then there is an iterated ultrapower $k : N \to N'$ such that N' is a $\mathcal{F}\restriction \kappa + 1$-mouse.

We will define $\mathcal{F}'$ recursively so that $\mathcal{F}'$ satisfies (1), (2) and (3′). Then we will use an argument using Proposition 2.21 to show that the definition of $\mathcal{F}'$ must stop at some ordinal κ. The construction will be such that it can only stop if $M \in Q$; hence we can conclude that, as claimed, $M \in Q$.

We start the definition of $\mathcal{F}'$ by setting $\mathcal{F}' \restriction \ell(\mathcal{F}) = \mathcal{F}$, so (1) holds. Now suppose $\kappa \geq \ell(\mathcal{F})$ and suppose $\mathcal{F}' \restriction \kappa$ has already been defined so that (2) and (3′) hold at all $\kappa' < \kappa$. We show how to define $\mathcal{F}'$ at κ. Let $j^\kappa : K(\mathcal{F}') \to Q^\kappa$ be the extension of j to $K(\mathcal{F}' \restriction \kappa)$ given in Definition 5.3(ii). Now let $M = J^{\mathcal{G}}_\alpha$ and use the technique of the proof of Theorem 3.3, Part 1, to find an ordinal ν and iterated ultrapowers of length ν

$$k^\kappa_\nu : Q^\kappa \to Q^\kappa_\nu$$

$$i^\kappa_\nu : M \to M^\kappa_\nu$$

so that if $\mathcal{F}^\kappa_\nu = k^\kappa_\nu(j^\kappa(\mathcal{F}' \restriction \kappa))$ and $\mathcal{G}^\kappa_\nu = i^\kappa_\nu(\mathcal{G})$ then either $\mathcal{F}^\kappa_\nu = \mathcal{G}^\kappa_\nu \restriction \ell(\mathcal{F}^\kappa_\nu)$ or else $\mathcal{G}^\kappa_\nu = \mathcal{F}^\kappa_\nu \restriction \ell(\mathcal{G}^\kappa_\nu)$.

We claim that if $\mathcal{G}^\kappa_\nu = \mathcal{F}^\kappa_\nu \restriction \ell(\mathcal{G}^\kappa_\nu)$ then $M \in Q$; in this case we can simply terminate the construction. If $\mathcal{G}^\kappa_\nu = \mathcal{F}^\kappa_\nu \restriction \ell(\mathcal{G}^\kappa_\nu)$ then $M = J^{\mathcal{G}^\kappa_\nu}_{\alpha_\nu} \in L(\mathcal{F}^\kappa_\nu)$, so $M \in Q^\kappa_\nu$. Then M, the transitive collapse of the Σ_1

skolem hull of $\delta \cup i^K_\nu(\rho_M)$ in M^K_ν, is also in Q^K_ν. But M can be coded by a subset of δ and $k^K_\nu \upharpoonright \delta = \mathrm{id}$ so $\mathcal{P}(\delta) \cap Q^K_\nu \subset Q^K$ and hence $M \in Q^K$. But $\mathcal{P}(\delta) \cap Q^K \subset Q$, so $M \in Q$.

For the rest of the proof we will assume $M \notin Q$. Thus $\mathcal{G}^K_\nu \upharpoonright i^K_\nu j^K(\kappa) = \mathcal{F}^K_\nu$. We set $\sigma^{\mathcal{F}'}(\kappa) = 0$ unless

(4) $j^K(\kappa) = k^K_\nu(\kappa) = \kappa = \nu$ and $a_\mu < \nu$ for all $\mu < \nu$.

Here (a_μ, b_μ) is, as in the proof of Part 1 of Theorem 3.3, the least pair at which $\mathcal{G}^K_\mu$ and $\mathcal{F}^K_\mu$ differ. If (4) holds then set $\mathcal{F}_\nu = \mathcal{F}^K_\nu$, $\mathcal{G}_\nu = \mathcal{G}^K_\nu$, and $j_\nu = k^K_\nu j^K$, and define $U(\nu,\beta) = \{x \in \mathcal{P}(\nu) \cap K(\mathcal{F}' \upharpoonright \kappa) : j_\nu(x) \in \mathcal{G}_\nu(\nu, j_K(\beta))\}$ for all β such that $j_\nu(\beta) < o^{\mathcal{G}_\nu}(\nu)$. We will set $\mathcal{F}'(\nu,\beta)$ equal to $U(\nu,\beta)$ for all β such that $j_K(\beta) < o^{\mathcal{G}_\nu}(\nu)$ and β satisfies (5) and (6) below. $o^{\mathcal{F}'}(\nu)$ is defined to be the least ordinal such that one of these conditions fails.

(5) $U(\nu,\beta)$ is a countably complete $K(\mathcal{F}' \upharpoonright (\nu,\beta))$ ultrafilter sequence.

(6) If $\ell(\mathcal{F}) \leq \kappa' < \kappa$ and N is a $\mathcal{F}' \upharpoonright (\kappa'+1)$-mouse with $|N| \leq (\kappa')^{++}$ in $K(\upharpoonright (\kappa'+1))$ then there is an iterated ultrapower $k : N \to N' = J^{\mathcal{H}}_\tau$ such that $\mathcal{H} \upharpoonright (\nu,\beta) = \mathcal{F}' \upharpoonright (\nu,\beta)$ and $\mathcal{H}(\nu,\beta) = U(\nu,\beta)$ or else $\mathcal{H} = \mathcal{F}' \upharpoonright \tau$ and $\tau < \nu$.

This completes the definition of $\mathcal{F}'$. We have seen that if $M \notin Q$ then the construction never terminates. We will complete the proof that $M \in Q$ and hence the proof of Lemma 5.5 by showing that the assumption $M \notin Q$ also implies that the construction terminates. This contradiction will show that $M \in Q$.

The proof will use the fact that the maps $i_\nu^\kappa j_\nu^\kappa$ and k_ν^κ are essentially independent of κ. Suppose $\kappa' < \kappa$; then $j^\kappa : K(\mathfrak{F}'\restriction\kappa) \to Q^\kappa$ is the extension to $K(\mathfrak{F}'\restriction\kappa)$ of $j^{\kappa'} : K(\mathfrak{F}'\restriction\kappa') \to Q^{\kappa'}$. In particular if κ' is regular in $K(\mathfrak{F}'\restriction\kappa')$ then $j^\kappa(\kappa') = j^{\kappa'}(\kappa')$ and Q^κ and $Q^{\kappa'}$ have the same subsets of δ for any $\delta < j^\kappa(\kappa')$. It follows that $i^\kappa_{\nu_{\kappa'}} = i^{\kappa'}_{\nu_{\kappa'}}$ and $k^\kappa_{\nu_{\kappa'}}$ is the extension of $k^{\kappa'}_{\nu_{\kappa'}}$ to Q^κ. Hence we can drop the superscript κ from i_ν^κ. We will also drop the superscript κ from j_ν^κ and k_ν^κ. To see what is going on here, define $j^* : K(\mathfrak{F}') \to Q^*$, $k_\nu : Q^* \to Q_\nu$, and $i_\nu : M \to M_\nu$ exactly like j^κ, k_ν^κ and i_ν^κ were defined from $K(\mathfrak{F}'\restriction\kappa)$. Then j^*, k_ν and i_ν agree with j^κ, k_ν^κ and i_ν^κ on the part of their domain which was used to define $\mathfrak{F}'(\nu,\lambda)$. Thus we can define $\mathfrak{F}'$ from j^*, k_ν and i_ν, and this is the definition which we will actually be using. We could not have used this as the primary definition, though, because it would have been circular.

Start with $\Gamma = \{\nu \in ON : cf(\nu) > \omega\}$. We will use Proposition 2.21 and related arguments repeatedly to shrink Γ to smaller stationary subclasses with special properties. Eventually these properties will be strong enough to conclude that Γ is empty, contradicting the fact that Γ is still stationary. This contradiction will show that our assumption that the construction never terminates is false and hence complete the proof of Lemma 5.5. Variables ν and ν' always range over all members of the current class Γ; thus the statement "$P(\nu,\nu')$" means that every pair $\nu,\nu' \in \Gamma$ (or, depending on the context, every pair ν,ν' with $\nu < \nu'$ or every pair with $\nu' < \nu$) has the property P.

<u>Claim 1</u>: Γ can be shrunk so that $i_{\nu'\nu}(\nu') = \nu$, $i_{\nu'\nu}(b_{\nu'}) = b_\nu$, and $a_\nu = \nu$.

<u>Proof</u>: We have $\nu \le a_\nu \in M_\nu$, $b_\nu \in M_\nu$ and M is a set so Proposition 2.21

allows Γ to be shrunk so that $i_{\nu'\nu}(\nu') = \nu$ and $i_{\nu'\nu}(b_{\nu'}) = b_\nu$. Now $\nu < a_\nu$ would imply $i_{\nu\nu'}(\nu) = \nu$ for $\nu' > \nu$ but $i_{\nu\nu'}(\nu) = \nu' > \nu$, so we must have $\nu = a_\nu$. □ Claim 1

Claim 2: Γ can be shrunk so that for some fixed μ and all $\nu \in \Gamma$, $k_{\mu\nu}(\nu) = \nu$.

Proof: There are $\mu_\nu < \nu$ and $\gamma_\nu \leq \nu$ so that $k_{\mu_\nu \nu}(\gamma_\nu) = \nu$. We can shrink Γ so that $\mu_\nu = \mu$ is constant. If $\gamma_\nu = \nu$ then $k_{\mu\nu}(\nu) = \nu$, so if the claim is false then we can shrink Γ so that $\gamma_\nu < \nu$ for all ν in Γ. We can shrink Γ further so that $\gamma_\nu = \gamma$ is constant and hence $k_{\nu'\nu}(\nu') = \nu$. In particular $k_{\nu\nu'}(\nu) > \nu$, so $b_\nu < o^{\mathcal{F}_\nu}(\nu)$. Since $i_{\nu\nu'}(\nu) = \nu' > \nu$, $b_\nu < o^{\mathcal{G}_\nu}(\nu)$ as well so there is $X_\nu \subset \nu$ such that $\mathcal{G}_\nu(\nu, b_\nu)$ and $\mathcal{F}_\nu(\nu, b_\nu)$ disagree on X_ν. As in Theorem 3.3 we can shrink Γ so that $k_{\nu\nu'}(X_\nu) = i_{\nu\nu'}(X_\nu) = X_{\nu'}$ and conclude that $X_\nu \in \mathcal{F}_\nu(\nu, b_\nu)$ iff $\nu \in X_{\nu'}$ iff $X_\nu \in \mathcal{G}_\nu(\nu, b_\nu)$, contrary to the choice of X_ν. □ Claim 2

Claim 3: Γ can be shrunk so that $j_\nu(\nu) = \nu$.

Proof: We will first show that there are fixed ordinals τ and η such that Γ can be shrunk so that for $\nu \in \Gamma$ there is $f_\nu : \tau \to \mathrm{ON}$ in $K(\mathcal{F}')$ such that $\nu = j_\nu(f_\nu)(\eta)$ and $j_\nu(\tau) < \nu$. Pick μ by Claim 2 so that $k_{\mu\nu}(\nu) = \nu$ for all $\nu \in \Gamma$. We can pick τ so that $a_\gamma < j_\mu(\tau)$ for all $\gamma < \mu$ and we can shrink Γ so that $\nu > j_\mu(\tau)$ for $\nu \in \Gamma$.

Now every ordinal in Q_ν has the form $j_\nu(f)(x)$ where $f \in K(\mathcal{F}')$ and x is a finite subset of $\delta \cup \{a_{\mu'} : \mu' < \mu\}$. In particular for each

ν in Γ we can code x by an ordinal and hence find $f_\nu \in K(\mathfrak{F}')$ and $\eta_\nu < \mu$ such that $\nu = j_\mu(f_\nu)(\eta_\nu)$. Since $\eta_\nu < \mu$ for all ν, we can shrink Γ so that $\eta_\nu = \eta$ is constant. But then $k_{\mu\nu}(\eta) = \eta$ for all ν, so $j_\nu(f_\nu)(\eta) = k_{\mu\nu}(j_\nu(f_\nu)(\eta)) = k_{\mu\nu}(\nu) = \nu$ and η, τ and f_ν are as required.

Now define $f^*_\nu = j_\nu(f_\nu) \upharpoonright \{\xi < j_\nu(\tau) : j_\nu(f_\nu)(\xi) < \nu\}$. We will show that Γ can be shrunk so that $f^*_{\nu'} = f^*_\nu$. We have $f^*_\nu \in Q_\nu$ and Q_ν satisfies the sentence "$V = K(\mathfrak{F}_\nu)$", so f^*_ν is in an $\mathfrak{F}_\nu \upharpoonright \nu$ mouse N in Q_ν. But M_ν is an $\mathfrak{F}_\nu \upharpoonright \nu$ mouse and by assumption $M_\nu \notin Q_\nu$. It follows that $N < M_\nu$, so $f^*_\nu \in M_\nu$. Since M is a set we can use Proposition 2.21 to shrink Γ so that $i_{\nu'\nu}(f^*_{\nu'}) = f^*_\nu$. But $i_{\nu'\nu} \upharpoonright \nu' = \mathrm{id}$ and $f^*_{\nu'} \subset \tau \times \nu'$. Since $\tau < \nu'$ it follows that $f^*_\nu = i_{\nu'\nu}(f^*_{\nu'}) = f^*_{\nu'}$.

Now take $\nu' < \nu$ in Γ and let β_ν be the least ordinal such that for some ordinal ξ, $\beta_\nu = f_\nu(\xi) \neq f_{\nu'}(\xi)$. Clearly $j_\nu(\beta_\nu) < \nu$, since $j_\nu(f_\nu)(\eta) = \nu$ and $j_\nu(f_{\nu'})(\eta) = k_{\nu'\nu}(j_{\nu'}(f_{\nu'})(\eta)) = k_{\nu'\nu}(\nu') < k_{\nu'\nu}(\nu) = \nu$. But $j_\nu(\beta_\nu) \nless \nu$, since otherwise we would have $f^*_\nu(\xi) = j_\nu(\beta_\nu) \neq f^*_{\nu'}(\xi)$. Hence $\nu = j_\nu(\beta_\nu)$. Now $\beta_\nu \leq \nu$, and $\beta_\nu \nless \nu$ since otherwise we could shrink Γ so that $\beta_\nu = \beta$ is constant and hence $\nu = k_{\nu'\nu}(\nu') < \nu$. Hence $\beta_\nu = \nu$ and so $j_\nu(\nu) = \nu$. $\square$ Claim 3

We now know that if $\nu \in \Gamma$ then $j_\nu(\nu) = \nu > \ell(\mathfrak{F})$ and $\nu = a^{\mathfrak{F}}_\nu > a^{\mathfrak{G}_\nu}_\mu$ for all $\mu < \nu$. Hence (4) is satisfied at ν. Also $j_\nu(o^{\mathfrak{F}'}(\nu)) = o^{\mathfrak{F}_\nu}(\nu) = b_\nu < o^{\mathfrak{G}_\nu}(\nu)$, so either (5) or (6) must fail for $\beta = \beta_\nu = o^{\mathfrak{F}'}(\nu)$. Set $W_\nu = U(\nu, \beta_\nu) = \{x \subset \nu : j_\nu(x) \in \mathfrak{G}_\nu(\nu, j_\nu(\beta_\nu))\}$.

<u>Claim 4</u>: Γ can be shrunk so that W_ν is countably complete.

Proof: Otherwise let $X_{\nu,n}$ be a sequence of sets in W_ν such that $\bigcap_{n\in\omega} X_{\nu,n} = 0$. Then, as with f^*_ν in Claim 3, $j_\nu(X_{\nu,n}) \in M_\nu$ for all $\nu \in \Gamma$ and $n \in \omega$ and we can shrink Γ so $i_{\nu'\nu}(j_{\nu'}(X_{\nu',n})) = j_\nu(X_{\nu,n})$ for all n. Since $X_{\nu',n} \in W_{\nu'}$, $j_{\nu'}(X_{\nu',n}) \in \mathcal{G}_{\nu'}(\nu',b_{\nu'})$ and hence $\nu' \in i_{\nu'\nu}(j_{\nu'}(X_{\nu',n})) = j_\nu(X_{\nu,n})$. But $j_{\nu'}(\nu') = \nu'$ and $k_{\nu'\gamma}(\nu') = \nu'$ for $\gamma > \nu'$, so $\nu' = j_\nu(\nu')$ and hence $\nu' \in \bigcap_{n\in\omega} X_{\nu,n}$, contrary to the choice of $X_{\nu,n}$. □ Claim 4

Claim 5: Γ can be shrunk so that W_ν is normal.

Proof: Otherwise let f_ν be such that $\{\eta: f_\nu(\eta) < \eta\} \in W_\nu$ but $\{\eta: f_\nu(\eta) = \gamma\} \notin W_\nu$ for all $\gamma < \nu$. Since $\mathcal{G}_\nu(\nu,b_\nu)$ is normal there is $\gamma_\nu < \nu$ such that $\{\eta: j_\nu(f_\nu)(\eta) = \gamma_\nu\} \in \mathcal{G}_\nu(\nu,b_\nu)$. Shrink Γ so that $\gamma_\nu = \gamma$ is constant and, as in Claim 3, so that $i_{\nu'\nu}(j_{\nu'}(f_{\nu'})) = j_\nu(f_\nu)$. Then $\gamma = j_\nu(f_\nu)(\nu') = j_\nu(f_\nu(\nu'))$ so if $\gamma' = f_\nu(\nu')$ then $\{\eta: f_\nu(\eta) = \gamma'\} \in W_\nu$, contrary to the choice of f_ν. □ Claim

Claim 6: Γ can be shrunk so that W_ν is a countably complete $K(\mathcal{F}'\restriction\nu+1)$ ultrafilter.

Proof: After Claims 4 and 5 we only need to prove coherence. We show first that if f is a function such that $\{\eta \in \nu: f(\eta) < o^{\mathcal{F}'}(\eta)\} \in W_\nu$ then there is $\gamma < o^{\mathcal{F}'}(\nu)$ such that

$$(7) \qquad \{\eta: j_\nu(f)(\eta) = C(\nu,j_\nu(\gamma),b_\nu(\eta))(\eta)\} \in \mathcal{G}_\nu(\nu,b_\nu).$$

(Note that the function C is computed in M_ν, using $\mathcal{G}_\nu(\nu,b_\nu)$.) Otherwise pick f_ν for each ν so that (7) fails. Then for some $\gamma_\nu < b_\nu = j_\nu(\beta_\nu)$ we have $\{\eta: j_\nu(f_\nu)(\eta) = C(\nu,\gamma_\nu,b_\nu)(\eta)\} \in \mathcal{G}_\nu(\nu,b_\nu)$. As in Claim 5, shrink

Γ so that $i_{\nu'\nu}(\gamma_{\nu'}) = \gamma_\nu$ and $i_{\nu'\nu}(j_{\nu'}(f_{\nu'})) = j_\nu(f_\nu)$. Then $\gamma_{\nu'} = j_{\nu'\nu}(j_{\nu'}(f_{\nu'})(\nu')) = j_\nu(f_\nu)(\nu') = j_\nu(f_\nu(\nu'))$. But $k_{\nu'\nu} \circ o^{\mathfrak{F}_{\nu'}}(\nu') = \mathrm{id}$ so $k_{\nu'\nu}(\gamma_{\nu'}) = \gamma_{\nu'}$ so $\gamma_{\nu'} = j_{\nu'}(f_\nu(\nu'))$ and (7) holds at ν' for $\gamma = f_\nu(\nu')$, contrary to the choice of $f_{\nu'}$.

It follows that if $\{\eta\colon f(\eta) < o^{\mathfrak{F}'}(\eta)\} \in W_\nu$ then there is $\gamma < o^{\mathfrak{F}'}(\nu)$ such that $[\lambda\eta\ \mathfrak{F}'(\eta, f(\eta))]_{W_\nu} = \mathfrak{F}'(\nu,\gamma)$. To complete the proof of coherence we have to show that Γ can be shrunk so that for each $\nu \in \Gamma$ and $\gamma < o^{\mathfrak{F}'}(\nu)$ there is an $f \in L(\mathfrak{F}' \restriction \nu+1)$ such that

(8) $$\{\eta\colon j_\nu(f)(\eta) = C(\nu, j_\nu(\gamma), b_\nu)(\eta)\} \in \mathcal{G}_\nu(\nu, b_\nu).$$

If not then Γ can be shrunk so that for each $\nu \in \Gamma$ there is $\gamma_\nu < o^{\mathfrak{F}'}(\nu)$ such that (8) is false for all $f \in L(\mathfrak{F}' \restriction \nu+1)$. Shrink Γ so that $i_{\nu'\nu}(j_{\nu'}(\gamma_{\nu'})) = j_\nu(\gamma_\nu)$ and let $f_\nu = C(\nu, j_\nu(\nu), b_\nu)$ in M. Then $i_{\nu'\nu}(f_{\nu'}) = f_\nu$ and $f_\nu(\nu') = i_{\nu'\nu}(f_{\nu'})(\nu') = i_{\nu'+1,\nu}(j_{\nu'}(\gamma_{\nu'}))$ $= j_{\nu'}(\gamma_{\nu'}) = j_\nu(\gamma_{\nu'})$. We have $f_\nu \in L(\mathcal{G}_\nu \restriction (\nu, b_\nu)) = j_\nu(L(\mathfrak{F}' \restriction \nu+1))$, and the range of j_ν is cofinal in both ν and $\nu^{+(Q_\nu)}$ so there is an $\eta_\nu < \nu$ and a function $\sigma_\nu\colon \eta_\nu \to ({}^\nu\nu \cap L(\mathfrak{F}' \restriction \nu+1))$ in $K(\mathfrak{F}')$ such that $f_\nu \in$ range $(j_\nu(\sigma_\nu))$. Let δ_ν be such that $f_\nu = j_\nu(\sigma_\nu)(\delta_\nu)$ and shrink Γ so that $\eta_\nu = \eta$ and $\delta_\nu = \delta$ are constant and $i_{\nu'\nu}(j_{\nu'}(\sigma_{\nu'})) = j_\nu(\sigma_\nu)$. Now it is true in Q_ν that there is $\delta < j_\nu(\eta)$ such that $(j_\nu(\sigma_\nu)(\delta)(j_\nu(\nu')) = j_\nu(\gamma_{\nu'})$ since $j_\nu(\nu') = \nu'$ and $j_\nu(\sigma_\nu)(\delta) = f_\nu$. It follows that it is true in $K(\mathfrak{F}')$ that there is $\tau < \eta$ such that $\sigma_\nu(\tau)(\nu') = \gamma_{\nu'}$. We claim that (8) holds at ν' for $f = \sigma_\nu(\tau)$, contrary to the choice of $\gamma_{\nu'}$. It is enough to show that $A = \{\eta\colon j_{\nu'}(f)(\eta) = f_{\nu'}(\eta)\} \in \mathcal{G}_\nu(\nu, b_\nu)$, and hence it is enough to show that $\nu' \in i_{\nu'\nu}(A)$, that is, that $i_{\nu'\nu}(j_{\nu'}(f))(\nu') = i_{\nu'\nu}(f_{\nu'})(\nu')$. But $i_{\nu'\nu}(f_{\nu'})(\nu') = j_{\nu'}(\gamma_{\nu'})$ and $i_{\nu'\nu}(j_{\nu'}(f))(\nu') = i_{\nu'\nu}(j_{\nu'}(\sigma_{\nu'}(\tau)))(\nu') = j_\nu(\sigma_\nu(\tau))(\nu') = j_\nu(\sigma_\nu(\tau)(\nu'))$

$= j_\nu(\gamma_{\nu'}) = j_{\nu'}(\gamma_{\nu'})$. Thus $i_{\nu'\nu}(j_{\nu'}(f))(\nu') = i_{\nu'\nu}(f_{\nu'})(\nu')$. ☐ Claim 6

Claim 7: Γ can be shrunk so that (6) holds for β_ν at all $\nu \in \Gamma$.

Proof: Otherwise shrink Γ so that (6) fails for all $\nu \in \Gamma$, and pick for $\nu \in \Gamma$ a $\kappa_\nu < \nu$ and a $\mathcal{F}'\restriction\kappa_\nu + 1$-mouse N_ν witnessing the failure of (6). Now shrink Γ so that $N_\nu = N$ and $\kappa_\nu = \kappa$ are constant. By using the technique of Theorem 3.3 to compare N with $\mathcal{F}'$ we can construct an iterated ultrapower r such that

$$r_\nu : N \to N^\nu = J_{\tau_\nu}^{\mathcal{H}_\nu}$$

and $\mathcal{H}_\nu\restriction(\nu,\beta_\nu) = \mathcal{F}'\restriction\nu+1$ and either $o^{\mathcal{H}_\nu}(\nu) = \beta_\nu$ or $\mathcal{H}_\nu(\nu,\beta_\nu) \neq W_\nu$. We can shrink Γ so that $k_{\nu'\nu}(\nu') = \nu > \nu'$ and hence $o^{\mathcal{H}_\nu}(\nu) > \beta_\nu$, so $\mathcal{H}_\nu = \mathcal{H}_\nu(\nu,\beta_\nu) \neq W_\nu$. Pick $X_\nu \in N^\nu \cap K(\mathcal{F}'\restriction\nu+1)$ so that $X_\nu \in H_\nu - W_\nu$, and shrink Γ so that $r_{\nu'\nu}(X_{\nu'}) = X_\nu$ and $i_{\nu'\nu}(j_{\nu'}(X_{\nu'})) = j_\nu(X_\nu)$. Then $\nu' \in k_{\nu'\nu}(X_{\nu'}) = X_\nu$, so $\nu' \in j_\nu(X_\nu) = i_{\nu'\nu}(j_{\nu'}(X_{\nu'}))$. But then $j_{\nu'}(X_{\nu'}) \in \mathcal{G}_{\nu'}(\nu', b_{\nu'})$ and hence $W_{\nu'} \in U_{\nu'}$, contrary to the choice of $X_{\nu'}$. ☐ Claim 7

We have now shrunk Γ to a stationary class such that for all ν in Γ (4) holds and (5) and (6) hold for $\beta = \beta_\nu$. But this contradicts the choice of $\beta_\nu = o^{\mathcal{F}'}(\nu)$, so our assumption that the process never stops must be false. ☐ 5.5

§6 The Covering Lemma

The covering lemma proved by Jensen for L [De-J] and Dodd and Jensen for K and L(U) [D-J] has the clear and elegant statement that under the proper assumptions the model M in question has the covering property: If x is any set of ordinals then there is a set y in M such that $x \subset y$ and $|x| = |y \cup x_1|$. Because the structure of indiscernibles in $K(\mathcal{F})$ is much more complex than in L(U) we do not know whether the covering property can be proved for $K(\mathcal{F})$. This problem will be discussed further in later papers; in this paper we restrict ourselves to the weak covering property:

<u>6.1 Definition</u>: M has the weak covering property if for every sufficiently large singular strong limit cardinal κ, $\kappa^{+(M)} = \kappa^{+}$.

<u>6.2 Theorem</u>: If there is no model of $\exists\kappa\, o(\kappa) = \kappa^{++}$ then there is a strong sequence $\mathcal{F}$ having the weak covering property. If $\mathcal{G}$ is any strong sequence with $\ell(\mathcal{G}) \in ON$ then $\mathcal{F}$ may be taken with $\mathcal{F}\restriction\ell(\mathcal{G}) = \mathcal{G}$.

Lemma 6.2 is proved by using the following stronger version of the covering lemma:

<u>6.3 Lemma</u>: Suppose that there is no model of $\exists\kappa\, o(\kappa) = \kappa^{++}$. Then for all ordinals μ and sets $A \subset \mu$ there is a sequence $\mathcal{F}$ with $\mathcal{F}\restriction\mu = 0$ which is strong in $K(\mathcal{F},A)$ and such that for all $\kappa > \mu$, if κ is regular in $K(\mathcal{F},A)$ and $(\mu \cup cf(\kappa))^{\aleph_0} < |\kappa|$ then there is a $K(\mathcal{F}\restriction\kappa,A)$-ultrafilter U on κ.

The assumption that $(\mu \cup cf(\kappa))^{\aleph_0} < |\kappa|$ can be weakened, with a little more care, to $(\mu \cup cf(\kappa) \cup \aleph_1) < |\kappa|$. However Lemma 6.3 will be adequate for our purposes.

<u>Proof of 6.2 from 6.3</u>: Let $A \subset \mu$ be a set coding up the sequence $\mathcal{G}$ and let $\mathcal{F}$ be as given by 6.2. Then since $\mathcal{F}$ is strong in $K(\mathcal{F},A) = K(\mathcal{F}\cup\mathcal{G})$, $K(\mathcal{F}\cup\mathcal{G}) \cap P(\mu) \subset K(\mathcal{G})$ so $\mathcal{G}$, and hence $\mathcal{F}\cup\mathcal{G}$, is strong. $\mathcal{F}\cup\mathcal{G}$ will be the desired sequence. Suppose that ν is a singular strong limit cardinal greater than μ and $\kappa = \nu^{+(K(\mathcal{F},A))} < \nu^{+}$. Then $\mathrm{cf}(\kappa) < \nu$ so $(\mathrm{cf}(\kappa)\cup\mu)^{\aleph_o} < \nu$. By Lemma 6.1 there is a $K(\mathcal{F}\restriction\kappa,A)$ ultrafilter on κ, but this is impossible because κ is a successor cardinal in $K(\mathcal{F}\restriction\kappa,A)$. $\square$ 6.2

<u>Proof of 6.3</u>: The sequence $\mathcal{F}$ is defined recursively. Set $\mathcal{F}\restriction\mu = 0$. If $\mathcal{F}\restriction(\alpha,\beta)$ has been defined and there is a countably complete $K(\mathcal{F}\restriction(\alpha,\beta),A)$ ultrafilter then pick any such ultrafilter for $\mathcal{F}(\alpha,\beta)$. Otherwise set $o^{\mathcal{F}}(\alpha) = \beta$. By (the relativization of) Lemma 4.1, $\mathcal{F}$ is strong in $K(\mathcal{F},A)$.

In the following we will for clarity simply ignore the set A. The presence of A has no affect on the proof except that obvious relativizations of earlier results are used.

Let κ be regular in $K(\mathcal{F})$ and suppose $(\mu\cup\mathrm{cf}(\kappa))^{\aleph_o} < |\kappa|$. Choose a cofinal subset z of κ such that $|z| = \mathrm{cf}(\kappa)$ and let X be an elementary substructure of $V_{\kappa+1}$, the sets of rank less than $\kappa+1$, such that

(1) $$|X| = (\mu\cup\mathrm{cf}(\kappa))^{\aleph_o},$$

(2) $$z \cup \{\mu,\mathcal{F}\restriction\kappa\} \cup \mu \subset X,$$

(3) $${}^{\omega}X \subset X.$$

Let $\pi:Q \cong X$ for a transitive set Q, and set $\bar{\kappa} = \pi^{-1}(\kappa)$ and $\bar{\mathfrak{F}} = \pi^{-1}(\mathfrak{F})$. The Proof of 6.3 breaks into two cases, depending on whether or not $K(\bar{\mathfrak{F}}) \cap P(\rho) \subset Q$ for all $\rho < \bar{\kappa}$.

<u>Case 1 ($K(\bar{\mathfrak{F}}) \cap P(\rho) \subset Q$ for all $\rho < \bar{\kappa}$)</u>. We will show that in this case there is a model of $\exists\kappa\ o(\kappa) = \kappa^{++}$. Under the hypothesis of the case $\bar{\mathfrak{F}}$ is strong, since it is strong in Q. Hence by Theorem 5.1 $K(\mathfrak{F}\restriction\delta) \subset K(\bar{\mathfrak{F}})$ for all $\delta \leq \bar{\kappa}$, so $K(\bar{\mathfrak{F}}\restriction\delta) \cap P(\delta) \subset Q$ for all $\delta < \bar{\kappa}$. Now fix δ equal to the least ordinal in $\kappa - X$. Then δ is the first ordinal moved by π, so $\bar{\mathfrak{F}}\restriction\delta = \mathfrak{F}\restriction\delta$. Also, $|\delta| = |X| < \kappa$ so $\delta^{+} < \kappa$ in $K(\mathfrak{F})$ since κ is a limit cardinal.

By definition 5.3(ii) the map $\pi:Q \longrightarrow X$ generates an extension $\pi^*:K(\mathfrak{F}) \longrightarrow (K(\mathfrak{F}))^*$. If $(K(\mathfrak{F}))^*$ is well founded then we identify it with a transition class.

<u>6.4 Lemma</u>: Suppose $\mathfrak{F}$ is strong, $\pi:Q \to B$ is an elementary embedding where B is a sufficiently large substructure of $K(\mathfrak{F})$, $K(\mathfrak{F}) \cap P(\delta) \subset Q$ where δ

is the first ordinal moved by π, and $(K(\mathfrak{F}))^*$ is well founded where $\pi^*: K(\mathfrak{F}) \to (K(\mathfrak{F}))^*$ is defined as above. If $o^{\mathfrak{F}}(\delta) < \delta^{++}$ in $L(\mathfrak{F}\upharpoonright\delta+1)$ then $U = \{x \subset \delta : \delta \in \pi(x)\}$ is a $K(\mathfrak{F}\upharpoonright\delta+1)$ ultrafilter on δ.

Proof: "Sufficiently Large" will be explained by the proof. In particular the π given in Case 1 works. By modification of Theorem 5.2, $(K(\mathfrak{F}))^* = K(\mathfrak{F}^*)$ for a strong sequence $\mathfrak{F}^*$. We first show that $\mathfrak{F}^*\upharpoonright\delta+1 = \mathfrak{F}\upharpoonright\delta+1$. Since δ is the first ordinal moved and $P(\delta) \cap K(\mathfrak{F}) \subset Q$, δ is a limit cardinal in $K(\mathfrak{F})$ and hence $\pi(\delta)$ is a limit cardinal in $K(\mathfrak{F})$. Thus $\pi(\delta) > \delta^{++(K(\mathfrak{F}))} > o^{\mathfrak{F}}(\delta)$. Any ordinal η less than $\pi(\delta)$ can be represented by the pair (id, η). The ordinal $o^{\mathfrak{F}^*}(\delta)$ can be represented by the pair $(\lambda\zeta < \delta(o^{\mathfrak{F}}(\zeta)), \delta)$ so $\eta = o^{\mathfrak{F}^*}(\delta)$ iff $(\delta, \eta) \in \pi(\{(\zeta, \zeta') : o^{\mathfrak{F}}(\zeta) = \zeta\})$; i.e., iff $\eta = o^{\mathfrak{F}}(\delta)$. Hence $o^{\mathfrak{F}}(\delta) = o^{\mathfrak{F}^*}(\delta)$.

If $\eta < o^{\mathfrak{F}}(\delta)$ then $\mathfrak{F}^*(\delta, \eta)$ is represented by the pair $((\lambda(\zeta_1, \zeta_2)\mathfrak{F}(\zeta_1, \zeta_2)), (\delta, \eta))$ and any subset x of δ is represented by the pair $(\lambda\zeta\, x \cap \zeta, \delta)$. Hence $x \in \mathfrak{F}^*(\delta, \eta)$ iff $(\delta, (\delta, \eta)) \in \pi(\{(\zeta, (\zeta_1, \zeta_2)) : x \cap \zeta \in \mathfrak{F}(\zeta_1, \zeta_2)\})$ iff $x \in \mathfrak{F}(\delta, \eta)$. Hence $\mathfrak{F}^*(\delta, \eta) = \mathfrak{F}(\delta, \eta)$ and since η was arbitrary $\mathfrak{F}^*\upharpoonright\delta+1 = \mathfrak{F}\upharpoonright\delta+1$.

Now we have an elementary embedding $\pi^*: K(\mathfrak{F}) \to K(\mathfrak{F}^*)$ with $\mathfrak{F}^*\upharpoonright\delta+1 = \mathfrak{F}\upharpoonright\delta+1$, and $U = \{x \subset \delta : \delta \in \pi^*(x)\}$. It is easy to see that U is normal. We will complete the proof by showing that if $o^{\mathfrak{F}}(\delta) \neq \delta^{++}$ in $L(\mathfrak{F}\upharpoonright\delta+1)$ then U is also coherent. Let $\mathfrak{H} = \mathfrak{F}\upharpoonright\delta+1 = \mathfrak{F}^*\upharpoonright\delta+1$ and consider the commutative triangle

$$\begin{tikzcd} L(\mathfrak{H}) \arrow[r, "i"] \arrow[dr, "\pi^*"'] & L(\mathfrak{H})^{\delta}/U = L(i(\mathfrak{H})) \arrow[d, "k"] \\ & L(\pi^*(\mathfrak{H})) \end{tikzcd}$$

where $k([f]) = \pi^*(f)(\delta)$ for any member $[f]$ of the ultrapower.

The claim that U is coherent in $K(\mathcal{F}\restriction\delta+1)$ translates straightforwardly to the claim that the first ordinal moved by k is greater than $o^{i(\mathcal{H})}(\delta)$, so that $o^{i(\mathcal{H})}(\delta) = o^{\mathcal{F}}(\delta)$. By assumption we have $o^{\mathcal{F}}(\delta) < \delta^{++}$ in $L(\mathcal{F}\restriction\delta+1) = L(\pi^*(\mathcal{H})\restriction\delta+1)$, so $o^{i(\mathcal{H})}(\delta) < \delta^{++}$ in $L(i(\mathcal{H})\restriction\delta+1)$. Hence it will be enough to show that the first ordinal moved by k is at least δ^{++} in $L(i(\mathcal{H})\restriction\delta+1)$. Now $k\restriction\delta+1$ is the identity, and if $\eta < \delta^+$ in $L(\mathcal{H})$ then there is a subset of δ of order type η in $L(\mathcal{H})$ and hence in $L(i(\mathcal{H}))$, so $k\restriction(\delta^{+(L(\mathcal{H}))})$ is the identity. If $\eta = \delta^+$ in $L(i(\mathcal{H})\restriction\delta+1)$ then $k(\eta) = \eta$ as well: Otherwise $\eta < k(\eta) = \delta^{+(L(\pi^*(\mathcal{H})\restriction\delta+1))}$ $= \delta^{+(L(\mathcal{H}))}$, so $\eta < \delta^{+(L(\mathcal{H}))}$ and $\eta = k(\eta)$, contrary to assumption. But the first ordinal moved by k is a cardinal in $L(i(\mathcal{H}))$ and hence in $L(i(\mathcal{H})\restriction\delta+1)$, so it must be at least δ^{++} in $L(i(\mathcal{H})\restriction\delta+1)$. $\square$ 6.4

<u>Proof of 6.3, continued</u>: We will show that if $\pi: Q \to V_{\kappa^+}$ is as defined before and $\pi^*: K(\mathcal{F}) \to (K(\mathcal{F}))^*$ then $(K(\mathcal{F}))^*$ is well founded. It then follows by Lemma 6.4 that the ultrafilter U is a $K(\mathcal{F}\restriction\delta+1)$ ultrafilter. But U is also countably complete: Otherwise let $(X_n : n \in \omega)$ be a sequence of sets in U such that $\bigcap_{n\in\omega} X_n = 0$. Since ${}^{\aleph_0}X \subset X$, $(X_n : n \in \omega) \in Q$ and hence $\bigcap_{n\in\omega} \pi(X_n) = \pi(\bigcap_{n\in\omega} X_n) = 0$. But this is impossible since $\delta \in \bigcap_{n\in\omega} \pi(X_n)$. Now if $\beta = o^{\mathcal{F}}(\delta)$ then U is a countably complete $K(\mathcal{F}\restriction(\delta,\beta))$-ultrapower, contradicting the definition of $o^{\mathcal{F}}(\delta)$.

If $(K(\mathcal{F}))^*$ is not well founded then let $((f_n,\eta_n): n \in \omega)$ be a sequence witnessing this, so if we set $x_n = \{(\eta,\eta'): f_{n+1}(\eta) \in f_n(\eta)\}$ then $(\eta_{n+1},\eta_n) \in \pi(x_n)$ for all n. The sequence $\vec{x} = (x_n : n \in \omega)$ is in Q, and $V_{\kappa+1} \models \exists \vec{\eta}(\forall n(\eta_{n+1},\eta_n) \in \pi(x_n))$, so $Q \models \exists \vec{\xi}(\forall n(\xi_{n+1},\xi_n) \in x_n)$. But then $(f_n(\xi_n) : n \in \omega)$ is a decreasing sequence of ordinals, which is impossible. $\square$ Case 1

Case 2 ($(K(\mathcal{F}) \cap P(\rho)) \not\subset Q$ for some $\rho < \bar{\kappa}$). In this case we will either show that κ is singular in $K(\mathcal{F})$ or else construct indiscernibles for $K(\mathcal{F})$ which can be used to define a $K(\mathcal{F})$-ultrafilter on κ. The first alternative is excluded by the hypothesis of the Lemma 6.3 we are trying to prove, so the $K(\mathcal{F}\restriction\kappa)$-ultrafilter required by the conclusion must exist. It should be remarked that this is the most interesting of the two possibilities, although this fact will not be apparent in the truncated version given here. In Case 1 larger cardinals exist than can be dealt with in $K(\mathcal{F})$, so the only information given by the argument is that the machinery is overwhelmed by reality. In Case 2, on the other hand, all large cardinals are in $K(\mathcal{F})$ and the proof gives quite a bit of information about how the universe of sets is built up from a base in $K(\mathcal{F})$.

If the hypothesis of Case 2 holds, then let M be the least $K(\mathcal{F})$-mouse such that $\mathcal{P}(\rho) \cap M \not\subset Q$ for some $\rho < \bar{\kappa}$. Then $M = J^{\mathcal{H}}_{\alpha+1}$ for some $\alpha \geq \bar{\kappa}$ and some sequence $\mathcal{H}$ with $\mathcal{H}\restriction\bar{\kappa}+1 = \bar{\mathcal{F}}$. We will carry out, as far as possible, a fine structure analysis of $J^{\mathcal{H}}_{\alpha}$ as in Section 4. Since $\mathcal{H}$ is an ultrafilter sequence above $\bar{\kappa}$, there is no problem for all n such that the projection ρ_n is not smaller than $\bar{\kappa}$. On the other hand there must be an n such that $\rho_{n+1} < \bar{\kappa}$ since by the minimality of M there is a subset of some $\rho < \bar{\kappa}$ in $J^{\mathcal{H}}_{\alpha+1} - J^{\mathcal{H}}_{\alpha}$. Hence there is a canonical Σ^*_n code $\mathfrak{A}_n = (M_n, A_n, \mathcal{H}\restriction\rho_n + 1)$ of $J^{\mathcal{H}}_{\alpha}$ such that $\rho_n = \cup A_n \geq \bar{\kappa}$ and $\mathfrak{A}_n$ has a new Σ^*_1 subset of some $\rho < \bar{\kappa}$. Now if $o^{\mathcal{H}}(\rho_n) > 0$ then, since $\rho_n \geq \bar{\kappa} \geq \ell(\bar{\mathcal{F}})$, $\mathcal{H}$ is a Σ_1 ultrafilter sequence and Σ_1 commutative in $\mathfrak{A}_n$. Hence by Lemma 4.41 $\mathfrak{A}_n$ is reducible to ρ via some parameter p, whether or not $o^{\mathcal{H}}(\rho_n) > 0$. Let $\mathcal{C}$ be the system of indiscernibles required for the reduction, so that $\mathfrak{A}_n = \Sigma_1^*(\rho \cup p;\mathcal{C})$. Let $\tau \geq \bar{\kappa}$ be the least ordinal above $\bar{\kappa}$ Σ_1^* definable in $\mathfrak{A}_n$ from $\rho \cup p$.

Claim: $\bigcup\{\mathcal{C}(\tau,\lambda) : \lambda < o^{\mathcal{H}}(\tau)\}$ is cofinal in $\bar{\kappa}$.

Proof: By the minimality of $J^{\mathcal{H}}_{\alpha+1}$, every set in $\mathfrak{A}_n$ is in Q and the map $\pi: Q \to V_{\kappa^+}$ defines a Σ_1^* elementary map $\pi^*: \mathfrak{A}_n \to \mathfrak{A}_n^*$ such that $\pi^* \restriction \bar{\kappa} = \pi \restriction \bar{\kappa}$. Since π^* is Σ_1^* elementary $\mathfrak{A}_n^*$ can be decoded to a structure $J^{\mathcal{H}^*}_{\alpha^*}$ such that $\mathcal{H}^* \restriction \kappa = \mathcal{F} \restriction \kappa$ and $\mathcal{H}^*$ has a rank ω complete system of indiscernibles above κ.

But then $\mathcal{H}^*$ is a ultrafilter sequence (above κ) in $\mathfrak{C} = J^{\mathcal{H}^*}_{\alpha^*+1}$. The Σ_1^* theory of $\mathfrak{A}_n^*$ is a member of $\mathfrak{C}$, so $\mathfrak{C}$ can be collapsed to give a $\mathcal{F} \restriction \kappa$-mouse containing the theory of $\mathfrak{A}_n^*$. It follows that this theory, and hence $\mathfrak{A}_n$ itself, is in $K(\mathcal{F} \restriction \kappa)$ and hence, by Theorem 5.1, is in $K(\mathcal{F})$.

Now suppose that $\bigcup\{\mathcal{C}(\tau,\lambda) \cap \bar{\kappa} : \lambda < o^{\mathcal{H}}(\tau)\}$ is bounded in $\bar{\kappa}$ and let δ be the sup of $\bigcup\{\mathcal{C}(\tau,\lambda) \cap \bar{\kappa} : \lambda < o^{\mathcal{H}}(\tau)\} \cup \rho$. Then the Σ_1^* hull of $\delta \cup p$ in $\mathfrak{A}_n$ is equal to $\mathfrak{A}_n$. Since $\pi''(\bar{\kappa})$ is cofinal in κ it follows that the Σ_1^* hull of $\pi''\delta \cup \pi(p)$ in $\mathfrak{A}^*$ is cofinal in κ. Then the Σ_1^* hull of $\pi(\delta) \cup \pi(p)$ is certainly cofinal in κ, but this set is in $K(\mathcal{F})$ and witnesses that κ is singular, contradicting the assumption that κ is regular in $K(\mathcal{F})$. $\square$ Claim

Now let $C = \bigcup\{\mathcal{C}(\tau,\lambda) \cap \bar{\kappa} : \lambda \subset o^{\mathcal{H}}(\tau)\}$ and for $c \in C$ define $Y_c = \{x \subset \kappa : \pi(c) \in x\}$ if $c \in \mathcal{C}(\tau,o)$ and $Y_c = \{x \subset \kappa : x \cap \pi(c) \in \mathcal{F}(\pi(c),o)\}$ if $c \in \mathcal{C}(\tau,\lambda)$ for some $\lambda > 0$. Set $U = \{x \subset \kappa : \exists\delta < \bar{\kappa}\ \forall c \in (C-\delta)\ x \in Y_c\}$. We claim that U is the $K(\mathcal{F} \restriction \kappa)$-ultrafilter on κ required by Lemma 6.3.

Suppose $f \in K(\mathfrak{F})$ and $\{\nu: f(\nu) < \nu\} \in U$ but for all $\gamma < \kappa$ $\{\nu: f(\nu) = \gamma\} \notin U$. For all sufficiently large $c \in C$ there is an ordinal $\gamma < \pi(c)$ such that $\{\nu: f(\nu) = \gamma\} \in Y_c$. Let's call this ordinal $f^*(c)$; then there is an increasing sequence $(c_n: n \in \omega)$ in C such that if $n \neq n'$ then $f^*(c_n) \neq f^*(c_n')$. Now since ${}^{\omega}Q \subset Q$, the sequence $(c_n: n \in \omega)$ is in Q and hence so is the sequence $(\overline{Y}_{c_n}: n \in \omega)$ where $\overline{Y}_{c_n} = \pi^{-1}(Y_{c_n})$.

Since π is Σ_1 elementary it is true in Q that there is a function $g \in K(\ )$ such that $\{\nu < c_n: g(\nu) < \nu\} \in \overline{Y}_{c_n}$ for each $n \in W$ but if γ_n is the ordinal such that $\{\nu < c_n: g(\nu) =\} \in \overline{Y}_{c_n}$ then $\gamma_n \neq \gamma_n$ whenever $n \neq n$. Since $g \in K(\mathfrak{F}) \cap Q$, g is in an $\mathfrak{F} \upharpoonright \kappa$-mouse in Q and hence $g \in J_\alpha$. Then g is in the Σ_1^* hull of $(p \cup p; \mathbb{C})$ in $\mathfrak{A}_n$ so $g = \tau(q, \vec{c})$ for some $q \subset \rho \cup p$, some sequence $\vec{c}$ and some Σ^* function τ. We can assume, by deleting an initial segment of $(c_n: n \in W)$ if necessary, that for each $c \in \vec{c}$ we have either $c < c_o$ or for each $n \in \omega$ $c > c_n$.

Now if there are integers $n < n'$ such that $c_n, c_{n'} \in (\tau, 0)$ then $\vec{c} \cup \{c_n\}$ is equivalent to $\vec{c} \cup \{c_{n'}\}$. Since $g(c_n) < c_n$ and $g(\nu) = \tau(q, \vec{c})(\nu)$ we have $\gamma_n = g(c_n) = g(c_{n'}) = \gamma_{n'}$, contrary to assumption. Similarly if there are integers $n < n'$ and a λ such that $c_n, c_{n'} \in \mathbb{C}(\tau, \lambda)$ then if $h(\nu) =$ the ordinal γ such that $\{\nu < \nu: g(\nu) = \gamma\} \in H(\nu, 0)$ then $h(\nu)$ is definable from $p \cup p \cup \vec{c} \cup \{\nu\}$ so $\gamma_n = h(c_n) = h(c_n = \gamma_n$, contrary to assumption. Finally, if there are itnegers $n < n'$ such that $c_n \in \mathbb{C}(\tau, \lambda)$ and $c_{n'} \in \mathbb{C}(\tau, \lambda')$ for ordinals $\lambda < \lambda'$ then $c_n \in \mathbb{C}(c_n, \bar{\lambda})$ where $\bar{\lambda} = C(\tau, \lambda, \lambda')(c_{n'})$. Then $\{\nu < c_n: g(\nu) = \gamma_n\} \in \mathcal{H}(c_n, 0)$ implies that $\{\eta \in c_{n'}: \{\nu < \eta: g(\nu) = \gamma_n\} \in \mathcal{H}(\eta, 0)\} \in \mathcal{H}(\gamma, \lambda)$. By coherence it follows that $\{\nu \in c_{n'}: g(\nu) = \gamma_n\} \in \mathcal{H}(c_n, 0)$ so again $\gamma_n = \gamma_{n'}$, contrary to assumption. But there must be some pair of integers $n < n'$ such that $c_n \in \mathbb{C}(\tau, \lambda)$, $c_n \in \mathbb{C}(\tau, \lambda')$ and $\lambda' \leq \lambda$, so the ordinals λ_n cannot all be distinct. $\square$

§7 Applications of the Weak Covering Property

In this section we will use the existence of sequences with the weak covering property to prove for $K(\mathfrak{F})$ some of the results which Kunen proved for $L(U)$ in [K 70] and which were extended to $L(\mathfrak{F})$, under a more restrictive hypothesis, in [M 74]. We will also show that these results are not always true for $L(\mathfrak{F})$.

All of the results of this section are proved without the axiom of choice beyond dependent choice. The only use which has been made so far of the full axiom of choice is in the inductive definition of the sequence $\mathfrak{F}$, in which we were required to choose an ultrafilter to be $\mathfrak{F}(\alpha,\beta)$ whenever possible. In Lemma 7.4 we will show that there is only one possible choice of $\mathfrak{F}(\alpha,\beta)$ and hence the axiom of choice is not required.

In Theorem 7.11 we will prove those results promised in the introduction that various hypotheses imply the existence of models satisfying $o(\kappa) = \kappa^{++}$.

In [K 70] frequent use is made of the fact that if Γ is a proper class then any subset in $L(U)$ of an ordinal α is definable in $L(U)$ from parameters in $\alpha \cup \Gamma$. This is true of $L(U)$ because the transitive collapse of the skolem hull of $\alpha \cup \Gamma$ in $L(U)$ contains all of the ordinals and hence must be $L(U')$ for some U'. In order to work in $K(\mathfrak{F})$ it is not enough to know that the transitive collapse of $H(\alpha \cup \Gamma)$ contains all the ordinals; it is also necessary to know that it contains all mice. The weak covering property will be used for this purpose.

<u>7.1 Definition</u>: A class Γ is τ-thick for a regular cardinal τ if Γ contains a τ-closed and unbounded subclass C such that $|\nu^+ \cap \Gamma| = \nu^+$

for all $\nu \in C$. A sequence $\mathfrak{F}$ is τ-full for a regular τ if there is τ-closed, unbounded class C of ordinals such that if $\nu \in C$ then ν^+ in $K(\mathfrak{F})$ is the same as in V.

A class is said to be thick, or a sequence to be full, if it is τ-thick or τ-full for all sufficiently large regular τ.

Note that Theorem 6.2 asserts that any strong sequence has an extension which is full.

If i is an iterated ultrapower of $K(\mathfrak{F})$ then we call i proper if the order type of $i(K(\mathfrak{F}))$ is at most ON or, equivalently, if no single ultrafilter is used ON many times in the ultrapower.

<u>7.2 Proposition</u>: Suppose $\mathfrak{F}$ is τ-full and Γ is τ-thick.

(i) Any intersection of τ-thick classes is τ-thick.

(ii) If i is a proper iterated ultrapower of $K(\mathfrak{F})$ then $i(\mathfrak{F})$ is full, and if $o^{\mathfrak{F}}(\nu) = 0$ whenever $cf(\nu) = \tau$ then $\{\nu : i(\nu) = \nu\}$ is τ thick.

(iii) If $\mathfrak{F}$ and $\mathfrak{G}$ are each τ-full then there is a τ-full sequence $\mathfrak{H}$ and iterated ultrapowers i and j:

$$\begin{array}{l} i: K(\mathfrak{F}) \searrow \\ \qquad\qquad\qquad K(\mathfrak{H}) \;. \\ j: K(\mathfrak{G}) \nearrow \end{array}$$

(iv) If M is isomorphic to the skolem hull $H(\Gamma \cup \gamma)$ of $\Gamma \cup \gamma$ in $K(\mathfrak{F})$ then $M = K(\mathfrak{H})$ for a τ-full sequence $\mathfrak{H}$.

(v) For all ordinals and sets $x \subset \alpha$ in $K(\mathfrak{F})$, x is definable from parameters in $\Gamma \cup \alpha$.

<u>Proof</u>: Clause (i) is clear and clause (ii) is clear unless the iteration has length ON. If i has length ON then suppose i is the limit of

$(i_\nu : \nu \in ON)$ and let X be the class of ordinals ν such that $(i_\nu(\nu) = \nu$ and $cf(\nu) = \tau$ and $\nu^{+(K(\mathfrak{F}))} = \nu^+)$. Then X contains a τ-closed unbounded class and for any $\nu \in C$ we have ν^+ in $K(i(\mathfrak{F}))$ equal to ν^+ in $K(\mathfrak{F})$, which is ν^+. If $o^{\mathfrak{F}}(\nu) = 0$ whenever $cf(\nu) = \tau$ then no member of X is measurable in $K(\mathfrak{F})$, so $i(\nu) = i_\nu(\nu) = \nu$ for ν in X. Thus $\{\nu : i(\nu) = \nu\}$ contains X, which is τ-thick.

To prove clause (iii), define the ultrapowers i and j to compare $\mathfrak{F}$ and $\mathfrak{G}$ as in Theorem 3.3. By the proof of Theorem 3.3 at least one of the classes $\{i_\nu(a) : \nu \in ON\}$ and $\{j_\nu(a) : \nu \in ON\}$ is bounded for each ordinal a. Hence at least one of i and j is proper; suppose i is proper. If j is also proper we are done, so suppose j is not proper, so $j(\mathfrak{G}) \restriction ON = i(\mathfrak{F})$. There is an ordinal ν_0 such that $C = \{\nu : j_{\nu_0 \nu}(a_{\nu_0}) = a_\nu = \nu > \nu_0\}$ is a closed and unbounded class. For all $\nu \in C$, ν^+ of $j(K(\mathfrak{G}))$ has real cardinality ν and so is less than the real ν^+. This is impossible, since $K(i(\mathfrak{F})) = K(j(\mathfrak{G} \restriction ON) \subset j(K(\mathfrak{G}))$ and $K(i(\mathfrak{F}))$ is τ-full by ii.

To prove clause (vi), let $\pi : M \cong \mathfrak{H}(\Gamma \cup \alpha) \prec K(\mathfrak{F})$. Then $M \models V = K(\mathfrak{G})$, where $\mathfrak{G} = \pi^{-1}(\mathfrak{F})$. There is a τ-closed class C of ordinals ν such that $cf(\nu) = \tau$, $|\nu \cap \Gamma| = \nu$, $\nu^+ = \nu^{+(K(\mathfrak{F}))}$, and $|\nu^+ \cap \Gamma| = \nu^+$. Then for $\nu \in C$ we have $\nu^{+(M)} = \nu^+$ so M must contain all of the $\mathfrak{G}$ mice on ν. Hence M is all of $K(\mathfrak{G})$ and $\mathfrak{G}$ is τ-full.

If Γ, $\mathfrak{F}$ and α are as in clause (v) then by (iv) we can take $\pi : K(\mathfrak{G}) \cong \mathfrak{H}(\Gamma \cup \alpha) \prec K(\mathfrak{F})$, with $\mathfrak{G}$ τ-full. Then by (iii) there are proper iterated ultrapowers $i : K(\mathfrak{F}) \to K(\mathfrak{H})$ and $j : K(\mathfrak{G}) \to K(\mathfrak{H})$. Then any subset x of α in $K(\mathfrak{F})$ is in $K(\mathfrak{H})$ and hence in $K(\mathfrak{G})$, so x is definable from parameters in $\Gamma \cup \alpha$. □

Using 7.2(ii) we can easily extend 6.2 to get:

<u>7.3 Lemma</u>: If $\mathfrak{G}$ is a strong sequence and $\ell(\mathfrak{G}) \in \mathrm{ON}$ then there is a strong, full $\mathfrak{F}$ such that $\mathfrak{F} \restriction \ell(\mathfrak{G}) = \mathfrak{G}$, every $\mathfrak{F}(\alpha,\beta)$ is countably complete, and $o^{\mathfrak{F}}(\alpha) = o$ for all α with $\mathrm{cf}(\alpha) > \ell(\mathfrak{G}) \cup \aleph_1$. □

<u>7.4 Lemma</u>: Suppose $\mathfrak{F}$ is a strong sequence, $\ell(\mathfrak{F}) \leq \kappa + 1$, and U_0 and U_1 are $K(\mathfrak{F})$-ultrafilters on κ. Then $U_0 = U_1$.

<u>Proof</u>: If $U_0 \neq U_1$ then the lemma also fails in $L(K(\mathfrak{F}), U_0, U_1)$, where the axiom of choice holds. Thus we can assume the axiom of choice. By Lemmas 6.4 and 7.3 there is a full sequence $\mathfrak{G}$ such that $\mathfrak{G} \restriction \kappa + 1 = \mathfrak{F}$ and $o^{\mathfrak{G}}(\nu) = 0$ whenever $\mathrm{cf}(\nu) > \kappa$. By Lemma 7.2 there are iterated ultrapowers

$$\begin{tikzcd} & \mathrm{ult}(K(\mathfrak{G}), U_0) \arrow[dr, "j_0"] & \\ K(\mathfrak{G}) \arrow[ur, "i_0"] \arrow[dr, "i_1"'] & & K(\mathfrak{H}) \\ & \mathrm{ult}(K(\mathfrak{G}), U_1) \arrow[ur, "j_1"'] & \end{tikzcd}$$

It is easy to see, using the fact that $o^{\mathfrak{G}}(\nu) = 0$ for all ν with $\mathrm{cf}(\nu) > \kappa$, that $\Gamma = \{\nu : j_0 i_0(\nu) = j_1 i_1(\nu) = \nu\}$ is thick. Also, $j_0(\kappa) = j_1(\kappa) = \kappa$ since $i_0(\mathfrak{G}) \restriction \kappa + 1 = j_0(\mathfrak{G}) \restriction \kappa + 1 = \mathfrak{F}$. By Proposition 7.2(v), any subset x of κ in $K(\mathfrak{G})$ is definable from parameters in $\kappa \cup \Gamma$. It follows that $j_0 i_0(x) = j_1 i_1(x)$. But $x \in U_0$ iff $\kappa \in i_0(x)$, which holds iff $\kappa \in j_0 i_0(x)$, and similarly for U_1. Thus $U_0 = U_1$. □

<u>7.5 Corollary</u>: If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are strong, and $o^{\mathfrak{F}_1}(\alpha) = o^{\mathfrak{F}_2}(\alpha)$ for all α then $\mathfrak{F}_1 = \mathfrak{F}_2$. □

This is not true if we look at $L(\mathfrak{F})$ instead of $K(\mathfrak{F})$. The following example answers a question left open in [M 74].

<u>7.6 Theorem</u>: Suppose κ is measurable and there are at least κ^+ measurable cardinals. Then there are distinct sequences $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of countably complete filters such that $o^{\mathfrak{F}_1}(\alpha) = o^{\mathfrak{F}_2}(\alpha) \le 1$ for all α and $\mathfrak{F}_i$ is an ultrafilter sequence in $L(\mathfrak{F}_i)$ for $i = 1,2$.

<u>Proof</u>: Since we are dealing with only one measure per cardinal, we will simplify our notation somewhat: Let $(a_\nu : \nu \subset a_o^+)$ be an increasing sequence of cardinals and let $\mathfrak{F}(a_\nu)$ be an ultrafilter on a_ν for $\nu < a_o^+$. We can assume that $V = L(\mathfrak{F})$.

<u>Claim</u>: For all $x \subset a_o$ there is $\eta < a_o^+$ such that $x \in L_{a_\eta}(\mathfrak{F} \restriction a_\eta)$.

<u>Proof</u>: Since $V = L(\mathfrak{F})$, every subset x of a_o is in an $\mathfrak{F} \restriction a_o + 1$-mouse N. Note that we can assume N is a "pet mouse" (see [M 79b]), that is, a model of ZF^-. Thus the Proof of Theorem 7.5 does not need the use of any of the machinery developed in this paper.

Now we compare the length of N with $L(\mathfrak{F})$, as in Theorem 3.3. Since we are working in $L(\mathfrak{F})$, where each $\mathfrak{F}(a_\nu)$ is a normal measure, this comparison will not affect $\mathfrak{F}$; that is, there is $\nu \le a_o^+$, an ordinal ξ, and an iterated ultrapower: such that $i: N \to L_\xi(\mathfrak{F}\ a_\nu)$ and $i \restriction a_o + 1 = \mathrm{id}$.

Let $N = J_\gamma^{\mathcal{G}}$. We can assume N satisfies that there are exactly a_o^+ measures. But $|a_o^{+(N)}| \le |N| = a_o$, so $a_o^{+(N)} < a_o^+$. But $a_o^{+(N)} = a_o^{+(i(N))}$, so $i(\mathcal{G}) = \mathfrak{F} \restriction a_\nu$ has only $a_o^{+(N)} < a_o^+$ measures. Thus $\nu = a_o^{+(N)} < a_o^+$. Now $\xi < a_\nu^+$, so if $\eta = \nu + 1$ then $x \in L_\xi(\mathfrak{F} \restriction a_\nu) \subset L_{a_\eta}(\mathfrak{F} \restriction a_\eta)$, as required. $\square$ claim

Now let Γ be the class of standard models M of cardinality a_o such that for some sequence $\mathfrak{F}_M$ of countably complete filters, M satisfies

(ZFC + $V = L(\mathcal{F}_M)$ +(the first $\mathcal{F}_M$-measurable cardinal is a_o)+($\ell(\mathcal{F}_M)$ is a successor ordinal)). For $M \in \Gamma$ let $F_M = \mathcal{F}_M(a_o) \cap L(\mathcal{F}_M\ a_M)$, where a_M is the largest $\mathcal{F}_M$-measurable cardinal. By the claim, $\mathcal{F}(a_o) \subset \cup\{F_M : M \in \Gamma\}$. Now $P(a_o) \subset K(\emptyset)$, so $\Gamma \in K(\emptyset)$. (Recall $K(\emptyset) = \bigcap_{\alpha \in ON} \text{ult}_\alpha(K(\mathcal{F}, \mathcal{F}(a_o)))$.) Since a_o is not measurable in $K(\emptyset)$, $\mathcal{F}(a_o)$ cannot be $\cup\{F_M : M \in \Gamma\}$. Thus $F_M \not\subset \mathcal{F}(a_o)$ for some $M \in \Gamma$, and there is a set $x \in L(\mathcal{F}_M \restriction a_M)$ such that $\mathcal{F}(a_o)$ and $\mathcal{F}_M(a_o)$ disagree on x. Now we can take an iterated ultrapower

$$i : N \to L_\xi(i(\mathcal{F}_M))$$

such that for some $\nu < a_o^+$, $i(\mathcal{F}_M \restriction a_M)$ is a sequence of measures in $L_\xi(i(\mathcal{F}_M))$ on the cardinals $(a_\gamma : \gamma < \nu)$. Since $L_\xi(i(\mathcal{F}_M))$ has the extra measure $i(\mathcal{F}_M(a_M))$, $i(\mathcal{F}_M \restriction a_M)$ is a sequence of measures in $L(i(\mathcal{F}_M \restriction a_M))$. But $x \in L(i(\mathcal{F}_M \restriction a_M))$ so $i(\mathcal{F}_M)(a_o) = \mathcal{F}_M(a_o) \neq \mathcal{F}(a_o)$. Then $\mathcal{F}_1 = \mathcal{F} \restriction a_\nu$ and $\mathcal{F}_2 = i(\mathcal{F}_M \restriction a_M)$ are the required sequences. $\square$ 7.6

On page 63 of [M 74] it was claimed that the conclusion of 3.4 of [M 74] always holds "except in finitely many places" in a sense which would imply in particular that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sequences such that $o^{\mathcal{F}_1}(\alpha) = o^{\mathcal{F}_2}(\alpha)$ for all α and there is $\mathcal{G}$ such that $L(\mathcal{G})$ is an iterated ultrapower of both $L(\mathcal{F}_1)$ and $L(\mathcal{F}_2)$ then $\mathcal{F}_1 = \mathcal{F}_2$. This claim is probably false; in any case we do not have a proof.

<u>7.7 Definition</u>: $\mathcal{F}$ is maximal at α if there is no strong sequence $\mathcal{F}'$ with $\mathcal{F}' \restriction \alpha = \mathcal{F} \restriction \alpha$ and $o^{\mathcal{F}'}(\alpha) > o^{\mathcal{F}}(\alpha)$.

Notice that if $\mathcal{F}'$ witnesses that $\mathcal{F}$ is not maximal then $\mathcal{F}' \restriction (\alpha, o^{\mathcal{F}}(\alpha)) = \mathcal{F} \restriction \alpha + 1$ and so $U = \mathcal{F}'(\alpha, o^{\mathcal{F}}(\alpha))$ is a $K(\mathcal{F} \restriction \alpha + 1)$ ultrafilter on α. The next lemma is of interest in itself but is given mainly for use in proving Theorem 7.9.

<u>7.8 Lemma</u>: Suppose that $\mathcal{F}$ is strong and $i:K(\mathcal{F}) \to K(\mathcal{F}')$ is an iterated ultrapower with support $\vec{f}$. Then for each α either

(i) $i(\mathcal{F})$ is maximal at α, or

(ii) $\alpha = i(\alpha')$ and $\mathcal{F}$ is not maximal at α', or

(iii) for some $\nu < \ell(\vec{f}) = \lambda$, $\alpha = [\lambda a\, a_\nu]^{\mathcal{F}'}_{\mathcal{F}(\vec{f})}$. Hence i factors into $i_{\nu+1,\lambda}\, i_{\nu,\nu+1}\, i_\nu$ where $\alpha = i_{\nu+1,\lambda}(\alpha')$, $o^{\mathcal{F}'}(\alpha) = i_{\nu+1,\lambda}(\beta')$, and $i_{\nu,\nu+1}$ is the ultrapower of $K(i_\nu(\mathcal{F}))$ by $i_\nu(\mathcal{F})(\alpha',\beta')$.

<u>Proof</u>: Suppose (i) fails and let U be a $K(\mathcal{F}' \restriction \alpha + 1)$-ultrafilter. Since the existence of U only depends on $\mathcal{F} \restriction i^{-1}(\alpha + 1)$ we can assume that $\mathcal{F}$ is full and that $\{cf(\gamma): o^{\mathcal{F}}(\gamma) > 0\}$ is bounded. By Lemma 7.2 there are maps

$$\begin{tikzcd} & & K(i(\mathcal{F}))^{\alpha}/U \arrow[dr, "k"] & \\ K(\mathcal{F}) \arrow[r, "i"] & K(i(\mathcal{F})) \arrow[ur, "j"] \arrow[rr, "r"] & & K(\mathcal{G}) \end{tikzcd}$$

such that if $\Gamma = \{\eta: i(\eta) = r(\eta) = j(\eta) = k(\eta) = \eta\}$ then Γ is thick.

<u>Case 1 ($\alpha \notin$ range (i))</u>. In this case we show that clause (iii) holds. If it does not then for some $f \in K(\mathcal{F})$ and $\gamma < \alpha$, $\alpha = i(f)(\gamma)$. But f is definable in $K(\mathcal{F})$ from members of $\Gamma \cup \bar{\alpha}$, where $\bar{\alpha} = \{\eta: i(\eta) < \alpha\}$, so α is definable in $K(i(\mathcal{F}))$ from members of $\Gamma \cup \alpha$. But $r \restriction (\Gamma \cup \alpha) = kj \restriction (\Gamma \cup \alpha)$ so $r(\alpha) = kj(\alpha)$, contradicting the fact that $\alpha = r(\alpha)$ and $\alpha < kj(\alpha)$.

Note that if clause (iii) holds then $i_{\nu+1,\lambda}(i_\nu(\mathcal{F})(\alpha',\beta'))$ satisfies the given conditions on U. Hence by Lemma 7.4 it must actually be equal to U.

<u>Case 2 ($\alpha = i(\alpha')$)</u>. Let $U' = \{x \subset \alpha : i(x) \in U\}$. We will show that U' is a $K(\mathfrak{F}\restriction(\alpha'+1))$-ultrafilter, so clause ii holds.

Suppose $f:\alpha' \to \alpha'$ and $\{\eta : f(\eta) < \eta\} \in U'$. Then $f = f'\restriction\alpha'$ for some f' definable from members of $\Gamma \cup \alpha'$. Then $r(i(f')) = kj(i(f'))$ and $kj(i(f))(\alpha) = i(f')(\alpha)$, so $\{\eta : i(f)(\eta) = i(f')(\alpha)\} \in U$. Hence $\{\eta : f(\eta) = f'(\alpha')\} \in U'$, and so U' is normal.

If $\{\eta : f(\eta) < o^{\mathfrak{F}}(\eta)\} \in U'$ then a similar argument shows that $[f]_{U'} = \beta'$ for some $\beta' < o^{\mathfrak{F}}(\alpha')$ and $[\lambda\eta\ \mathfrak{F}(\eta, f(\eta))]_{U'} = \mathfrak{F}(\alpha', \beta')$, so half of the coherence condition holds. For the other half, suppose $\beta' < o^{\mathfrak{F}}(\alpha')$. We have to show that there is $f' \in L(\mathfrak{F}\restriction\alpha'+1)$ such that $\beta' = [f']_{U'}$. Now since U is coherent and $\beta = i(\beta') < o^{i(\mathfrak{F})}(\alpha)$, there is $f \in L(i(\mathfrak{F})\restriction\alpha+1)$ such that $\beta = [f]_U$, and f is definable in $K(i(\mathfrak{F}))$ from parameters in $\Gamma \cup \alpha$. Let φ be a formula and let $\vec{x} \in [\alpha]^{<\omega}$ and $\vec{c} \in [\Gamma]^{<\omega}$ be parameters such that for all $\nu < \alpha$, $f(\nu) = \eta$ iff $\varphi(\vec{c}, \vec{x}, \nu, \eta)$. Then it is true in $K(i(\mathfrak{F}))$ that there exists $\vec{x} \in [\alpha]^{<\omega}$ such that if the function f is defined by $f(\nu) = \eta$ iff $\varphi(\vec{c}, \vec{x}, \nu, \eta)$ then $f\restriction\alpha \in L(i(\mathfrak{F})\restriction\alpha+1)$ and $f(\alpha) = \beta$. Since $i(\gamma) = \gamma$, it follows that in $K(\mathfrak{F})$ there is $\vec{x}' \in [\alpha']^{<\omega}$ such that if f' is defined by $f'(\nu) = \eta$ iff $\varphi(\vec{c}, \vec{x}', \nu, \eta)$ then $f'\restriction\alpha' \in L(\mathfrak{F}\restriction\alpha'+1)$ and $f'(\alpha') = \beta'$. But then $[f'\restriction\alpha']_{U'} = \beta'$, so U' is coherent.

Finally, $K(\mathfrak{F})^{\alpha'}/U'$ is well founded because it can be embedded in $K(i(\mathfrak{F}))^{\alpha'}/U'$, which is well founded. Since $\mathfrak{F}$ is full it follows from 7.2(iii) that U is absolutely well founded. $\square$

In Section 4 we proved Theorem 3.11 under the added hypothesis that $\mathfrak{F}(\alpha,\beta)$ is countably complete for each pair (α,β). In the next two results we eliminate this added hypothesis. A sequence $\mathfrak{F}$ is said to be <u>maximal</u> if it is maximal at all ordinals α, that is, for no α is there a $K(\mathfrak{F}\restriction(\alpha+1))$-ultrafilter.

<u>7.9 Theorem</u>: There is a maximal sequence $\mathfrak{F}_M$; this sequence is strong and is unique.

<u>Proof</u>: The existence of $\mathfrak{F}_M$ is an easy recursion on pairs (α,β): If $\mathfrak{F}_M\restriction(\alpha,\beta)$ is defined then $\mathfrak{F}_M(\alpha,\beta)$ is any $K(\mathfrak{F}_M\restriction(\alpha,\beta))$ ultrafilter, if any exists, and otherwise $o^{\mathfrak{F}}(\alpha)$ is set equal to β. By Lemma 7.4 the sequence is unique provided it is strong. Lemma 7.4 also implies that the axiom of choice is not needed for this construction (again, provided the sequence is strong). In order to show that $\mathfrak{F}_M$ is strong we will construct a sequence $\mathfrak{G}$ which is strong by Lemma 4.1 and then show that there is an elementary embedding of $K(\mathfrak{F}_M)$ into $K(\mathfrak{G})$ taking $\mathfrak{F}_M$ into $\mathfrak{G}$.

<u>Claim</u>: There is a full sequence $\mathfrak{G}$ such that for all α, if $o^{\mathfrak{F}}(\alpha) > 0$ then $cf(\alpha) = \omega_1$ and there is an ω_1-closed, unbounded class C of cardinals α such that $\mathfrak{G}$ is maximal at α.

<u>Proof</u>: Start with a sequence $\mathfrak{G}_1$ which is maximal for countably complete ultrafilters; that is, such that there is no countably complete $K(\mathfrak{G}_1\restriction\alpha+1)$-ultrafilter for any ordinal α. Then for any $\alpha > \omega_1$ with cofinality equal to ω_1 we have $\alpha^{+(K(\mathfrak{G}_1))} = \alpha^+$, so both α and α^+ have cofinality greater than ω. It follows that any $K(\mathfrak{G}_1\restriction\alpha+1)$ ultrafilter is countably complete, and hence $\mathfrak{G}_1$ is maximal at such α. Now take an iterated ultrapower $i\colon K(\mathfrak{G}_1) \to K(\mathfrak{G})$ as in Lemma 7.3 so that $cf(\alpha) = \omega_1$ for all α such that $o^{\mathfrak{G}}(\alpha) > 0$. Let C be the class of cardinals α such that $\alpha > \omega_1$, $cf(\alpha) = \omega_1$, and $i(\lambda) < \alpha$ for all $\lambda < \alpha$. Then C is ω_1 closed and unbounded; we will

show that $\mathcal{G}$ is maximal at all $\alpha \in C$. If $i(\alpha) = \alpha$ then $\mathcal{G}$ is maximal at α by Lemma 7.8, since $\mathcal{G}_1$ is maximal at α. If $i(\alpha) \neq \alpha$ then α is not in the range of i, so by Lemma 7.8 $\mathcal{G}$ is maximal at α unless i includes an ultrapower by an ultrafilter on α. But i doesn't include such an ultrapower, since $cf(\alpha) = \omega_1$ and i was constructed by taking ultrapowers on ordinals with cofinality different from ω_1. □ Claim

We will prove that $K(\mathcal{F}_M)$ is an elementary substructure of $K(\mathcal{G})$ by defining a class Z of ordinals such that $K(\mathcal{F}_M)$ is isomorphic to the skolem hull of Z in $K(\mathcal{G})$. Let z_α be the first α members of Z. The set z_α will be defined by induction on α together with a class Δ_α of ordinals such that $Z - z_\alpha \subset \Delta_\alpha$. These will satisfy the following 5 conditions:

(1) $\mathcal{H}(z_\alpha \cup \Delta_\alpha) \cap ON = z_\alpha \cup \Delta_\alpha$ where $\mathcal{H}(z_\alpha \cup \Delta_\alpha)$ is the skolem hull of $z_\alpha \cup \Delta_\alpha$.

(2) Δ_α is thick

(3) the order type of z_α is α

(4) If $\alpha' < \alpha$ then $z_{\alpha'} = z_\alpha \cap a_{\alpha'}$, where $a_{\alpha'} = \cap \Delta_{\alpha'}$, and $z_\alpha - z_{\alpha'} \subset \Delta_{\alpha'}$.

(5) If $\pi_\alpha : K(\mathcal{G}_\alpha) \cong \mathcal{H}(z_\alpha \cup \Delta_\alpha) \prec K(\mathcal{G})$ is the transitive collapse then $\mathcal{G}_\alpha \restriction \alpha = \mathcal{F}_M \restriction \alpha$. (Note that $\pi_\alpha(\alpha) = a_\alpha$ by (1) and (3).)

These conditions ensure that $K(\mathcal{F}_M) \cong \mathcal{H}(Z) \prec K(\mathcal{G})$, so $\mathcal{F}_M$ is strong.

We set $z_0 = 0$ and $\Delta_0 = ON$. If $z_{\alpha'}$ and $\Delta_{\alpha'}$ have been defined, $o^{\mathcal{F}_M}(\alpha') = 0$, and α is the least ordinal greater than α' such that $o^{\mathcal{F}_M}(\alpha) > 0$ then we can define z_α and Δ_α by letting z_α be the first α members of $z_{\alpha'} \cup \Delta_{\alpha'}$ and setting Δ_α equal to $\Delta_{\alpha'} - z_\alpha$. Conditions (1) - (4) are immediate, and $\mathcal{G}_\alpha \restriction \alpha' = \mathcal{G}_{\alpha'} \restriction \alpha' = \mathcal{F}_M \restriction \alpha'$. If $\alpha' \le \alpha'' < \alpha$ then $o^{\mathcal{F}_M}(\alpha'') = 0$, and $o^{\mathcal{G}_\alpha}(\alpha'') = 0$ as well by the maximality of $\mathcal{F}_M$. Hence $\mathcal{F}_M \restriction \alpha = \mathcal{G}_\alpha \restriction \alpha$ and condition (5) is satisfied. If α is a limit of measurable cardinals then we can set $z_\alpha = \bigcup_{\alpha'<\alpha} z_{\alpha'}$ and $\Delta_\alpha = \bigcap_{\alpha'<\alpha} \Delta_{\alpha'}$. Now we are left with the only difficult case, defining $z_{\alpha+1}$ and $\Delta_{\alpha+1}$ when z_α and Δ_α have been defined and $o^{\mathcal{F}_M}(\alpha) > 0$. To deal with this case we will define an auxiliary decreasing sequence of classes Γ_ν. For each ν conditions (1) - (5) will hold with Δ_α replaced by Γ_ν. In addition, if b_ν is the least member of Γ_ν then

(6) if $\nu' < \nu$ then $b_{\nu'} < b_\nu$.

If $\nu = 0$ then we set Γ_0 equal to Δ_α, and if ν is a limit ordinal then $\Gamma_\nu = \bigcap_{\nu'<\nu} \Gamma_{\nu'}$. We are left with the problem of defining $\Gamma_{\nu+1}$, given Γ_ν. For each ν let

$$\rho_\nu : K(\mathcal{G}'_\nu) \cong \mathcal{H}(z_\alpha \cup \Gamma_\nu) \prec K(\mathcal{G})$$

be the transitive collapse. Then $\mathcal{G}'_\nu \restriction \alpha = \mathcal{G}_\alpha \restriction \alpha = \mathcal{F}_M \restriction \alpha$. Also, by Lemma 7.4 $\mathcal{G}'_\nu(\alpha,\lambda) = \mathcal{F}_M(\alpha,\lambda)$ for any $\lambda < o^{\mathcal{G}'_\nu}(\alpha) \cap o^{\mathcal{F}_M}(\alpha)$. We must have $o^{\mathcal{G}'_\nu}(\alpha) = o^{\mathcal{F}_M}(\alpha)$ so $\mathcal{G}'_\nu \restriction \alpha+1 = \mathcal{F}_M \restriction (\alpha, o^{\mathcal{G}'_\nu}(\alpha))$. If $o^{\mathcal{G}'_\nu}(\alpha) = o^{\mathcal{F}_M}(\alpha)$ then we can set $z_{\alpha+1} = z_\alpha \cup \{b_\nu\}$ and $\Delta_{\alpha+1} = \Gamma_\nu - \{b_\nu\}$. We will eventually show that there always exists an ordinal ν such that $o^{\mathcal{G}'_\nu}(\alpha) = o^{\mathcal{F}_M}(\alpha)$, but first we will assume that $o^{\mathcal{G}'_\nu}(\alpha) = \beta_\nu < o^{\mathcal{F}_M}(\alpha)$ and we will construct $\Gamma_{\nu+1}$.

The filter $U = \mathcal{F}_M(\alpha,\beta_\nu)$ is a $K(\mathcal{G}'_\nu)$-ultrafilter on α, and $\mathrm{ult}(K(\mathcal{G}'_\nu),U)$ is well founded since U is absolutely well founded (Definition 3.10). By Lemma 7.2 we can construct proper iterated ultrapowers j and k so that the diagram

$$\begin{tikzcd}
 & \mathrm{ult}(K(\mathcal{G}_\nu'),\alpha) \arrow[dr, "k"] & \\
K(\mathcal{G}_\nu') \arrow[ur, "i"] \arrow[rr, "j"'] & & K(\mathcal{H})
\end{tikzcd}$$

commutes. Since Γ_ν was thick and $o^{\mathcal{G}}(\gamma) = 0$ if $\mathrm{cf}(\gamma) \neq \omega_1$, $Y = \{\eta : ki(\eta) = j(\eta)\}$ is thick. Now set $\Gamma_{\nu+1} = \rho_\nu''Y - b_\nu$. $\Gamma_{\nu+1}$ is thick since both Γ_ν and Y are and clearly $\mathcal{H}(\Gamma_{\nu+1} \cup z_\alpha) = \Gamma_{\nu+1} \cup z_\alpha$. Clauses (4) and (5) hold for $\Gamma_{\nu+1}$ because they hold for Γ_ν. Clause (6) is obvious: $\alpha \notin Y$ since $j(\alpha) = \alpha < i(\alpha)$, so $b_\nu = \rho_\nu(\alpha) \notin \Gamma_{\nu+1}$ and $b_\nu < b_{\nu+1}$. Hence $\Gamma_{\nu+1}$ satisfies conditions (1) - (6), and this completes the definition of the sequence of classes Γ_ν.

Since C, the class of cardinals of cofinality ω_1 where $\mathcal{G}$ is maximal, is ω_1-closed and unbounded there must be an ordinal $\nu \in C$ such that $b_{\nu'} < \nu$ for all $\nu' < \nu$. We will show that the construction of the Γ sequence must stop with Γ_ν.

Claim: Let ν be as above and let Γ be any thick class. Then $\nu \in \mathcal{H}(\nu \cup \Gamma)$.

Proof: Let $Q = K(\mathcal{G}')$ be the transitive collapse of $\mathcal{H}(\nu \cup \Gamma)$, so $\rho : Q \cong \mathcal{H}(\nu \cup \Gamma) \prec K(\mathcal{G})$. Suppose that, contrary to the claim, $\nu \notin \mathcal{H}(\nu \cup \Gamma)$. Then ν is the first ordinal moved by ρ. Since Γ is thick, $P(\nu) \cap K(\mathcal{G}) \subset Q$ so we can define $\rho^*: K(\mathcal{G}) \to (K(\mathcal{G}))^*$. If $(K(\mathcal{G}))^*$ is well founded then Lemma 6.4 implies that either $o^{\mathcal{G}}(\nu) = \nu^{++}$ in $K(\mathcal{G} \restriction \nu+1)$ or else $\mathcal{G}$ is not maximal at ν. But by assumption there is no model of $\exists\kappa\ o(\kappa) = \kappa^{++}$ and $\mathcal{G}$ is maximal at ν so $(K(\mathcal{G}))^*$ must not be well founded. We will complete the proof of the claim by showing that $(K(\mathcal{G}))^*$ really is well founded.

Suppose $(K(\mathcal{G}))^*$ is not well founded. Then there is a sequence $(f_n : n \in \omega)$ of functions $f_n : \nu \to \mathrm{ON}$ in $K(\mathcal{G})$ and a sequence $(\xi_n : n \in \omega)$ of ordinals $\xi_n < \rho(\nu)$ such that if $x_n = \{(\xi,\xi') : f_{n+1}(\xi') \in f_n(\xi)\}$ then $(\xi_n, \xi_{n+1}) \in \rho(x_n)$ for all $n \in \omega$. Now since $\mathcal{G}'$ is full there are, by Lemma 7.2, iterated ultrapowers i and j mapping $K(\mathcal{G}')$ and $K(\mathcal{G})$ into $K(\mathcal{H})$ for some sequence $\mathcal{H}$:

$$\begin{tikzcd} K(\mathcal{G}') \arrow[dr, "i"] & \\ & K(\mathcal{H}) \\ K(\mathcal{G}) \arrow[ur, "j"'] & \end{tikzcd}$$

Let $g_n \in K(\mathcal{G}')$ be such that $[g_n] = j(f_n)$, where the brackets represent the equivalence class in the ultrapower i. Then for each n we have

$$\rho(x_n) = \rho(\{(\xi,\xi') \in \nu^2 : \{\vec{a} : g_{n+1}(\vec{a})(\xi') \in g_n(\vec{a})(\xi)\} \in \mathcal{G}'\})$$

so

$$\rho(x_n) = \{(\xi,\xi') \in j(\nu^2) : \{\vec{a} : \rho(g_{n+1})(\vec{a})(\xi') \in \rho(g_n)(\vec{a})(\xi)\} \in \mathcal{G}\}.$$

(Note that to simplify the notation the support of the ultrapower has been omitted.) But this says that if $i' : K(\mathcal{G}) \to M$ is the ultrapower of $K(\mathcal{G})$ whose support is the image of the support of i then $([\rho(g_n)](i'(\xi_n)) : n \in \omega)$ is a decreasing sequence of ordinals in M. But M, an iterated ultrapower of $K(\mathcal{G})$, is well founded. $\square$ Claim

To complete the proof of Theorem 7.9 we will need one more claim.

Claim: If $\nu_1 < \nu_2$ then $(\mathcal{H}(b_{\nu_1} \cup \Gamma_{\nu_2}) \cap b_{\nu_2}) \subset b_{\nu_1}$.

Proof: If this fails then there is $\vec{w} \in [\Gamma_{\nu_2}]^{<\omega}$ and a term τ such that $K(\mathcal{G}) \models \exists\vec{x},y(\vec{x} \in [b_{\nu_1}]^{<\omega}$ and $b_{\nu_1} < y < b_{\nu_2}$ and $y = \tau(\vec{x},\vec{w}))$. Then $K(\mathcal{G}'_{\nu_1}) \models \exists\vec{x},y(\vec{x} \in [\alpha]^{<\omega}$ and $\alpha < y < \rho_{\nu_1}^{-1}(b_{\nu_2})$ and $y = \tau(\vec{x},\rho_{\nu_1}^{-1}(\vec{w}))$. Fix such an $\vec{x}$ and y. Then $\rho_{\nu_1}(\vec{x}) \subset z_\alpha$ and $K(\mathcal{G}) \models \rho_{\nu_1}(y) = \tau(\rho_{\nu_1}(\vec{x}),\vec{w})$ so $\rho_{\nu_1}(y) \in \mathcal{H}(z_\alpha \cup \Gamma_{\nu_2}) = z_\alpha \cup \Gamma_{\nu_2}$. But $z_\alpha < b_{\nu_1} < \rho_{\nu_1}(y) < b_{\nu_2} = \cap\Gamma_{\nu_2}$, so this is impossible. □ Claim

Now by the first claim with $\Gamma = \Gamma_\nu$ we must have $\nu \in \mathcal{H}(\nu \cup \Gamma_\nu)$, so $\nu \in \mathcal{H}(b_{\nu'} \cup \Gamma_\nu)$ for some $\nu' < \nu$. By the second claim it follows that $\nu \geq b_\nu$ and since $\nu = \bigcup_{\nu'<\nu} b_{\nu'}$ we must have $\nu = b_\nu$. Now if the construction does not stop at ν then $\Gamma_{\nu+1}$ is defined and $b_{\nu+1} > b_\nu = \nu$. By the second claim $b_\nu \notin \mathcal{H}(\nu \cup \Gamma_{\nu+1})$, but this contradicts the first claim. □ 7.9.

We could easily modify this argument to prove that $\mathcal{F}_M$ is full, but that is obvious from the next theorem:

7.10 Theorem: If $\mathcal{F}$ is any strong full sequence then there is an iterated ultrapower $i: K(\mathcal{F}_M) \to K(\mathcal{F})$. If $\mathcal{F}$ is strong but not full then there is an improper iterated ultrapower $i: K(\mathcal{F}_M) \to M$ such that $\mathcal{F} = i(\mathcal{F}_M) \upharpoonright \mathrm{ON}$.

Note that if $\mathcal{F}$ is a set then the second alternative can be restated: there is an iterated ultrapower $i: K(\mathcal{F}_M) \to K(i(\mathcal{F}_M))$ such that $\mathcal{F} = i(\mathcal{F}_M) \upharpoonright \ell(\mathcal{F})$.

Proof of 7.10: The ultrapower i is defined recursively: Suppose $i_\nu: K(\mathcal{F}_M) \to K(\mathcal{F}_\nu)$ has been defined. If $\mathcal{F}_\nu = \mathcal{F}$ then $i = i_\nu$. Otherwise by Lemma 7.4 there must be a_ν such that $o^{\mathcal{F}_\nu}(a_\nu) \neq o^{\mathcal{F}}(a_\nu)$ and if $b_\nu = \inf(o^{\mathcal{F}_\nu}(a_\nu), o^{\mathcal{F}}(a_\nu))$ then $\mathcal{F}_\nu \upharpoonright (a_\nu, b_\nu) = \mathcal{F} \upharpoonright (a_\nu, b_\nu)$. Now if

$o^{\mathfrak{F}_\nu}(a_\nu) < o^{\mathfrak{F}}(a)$ then $\mathfrak{F}_\nu$ is not maximal at a_ν. Since $\mathfrak{F}_M$ is maximal, case (iii) of Lemma 7.8 must hold, but this is impossible: it implies that $a_\nu = i_{\nu'+1,\nu}(a_{\nu'})$ for some $\nu' < \nu$ but by the construction $i_{\nu'+1,\nu}(a_{\nu'}) = a_{\nu'} < a_\nu$. Hence $b_\nu = o^{\mathfrak{F}_\nu}(a_\nu) > o^{\mathfrak{F}}(a_\nu)$ and $i_{\nu,\nu+1}$ is defined to be the ultrapower by $\mathfrak{F}_\nu(a_\nu, b_\nu)$. It is easy to see that i is an iterated ultrapower such that $i: K(\mathfrak{F}_M) \to K(\mathfrak{F})$ if i is proper and $i(\mathfrak{F}_M) \restriction ON = \mathfrak{F}$ if i is improper. If $\mathfrak{F}$ is full then i cannot be improper, so the first sentence of the theorem is true. Since there does exist a full sequence it follows that $\mathfrak{F}_M$ is full. Hence if $\mathfrak{F}$ is not full i cannot be proper, as the second sentence of the theorem states. □

Remark: Theorems 7.9 and 7.10 still hold if for some δ we make $\mathfrak{F}_M$ maximal only at cardinals α such that $cf(\alpha) < \delta$ and set $o^{\mathfrak{F}_M}(\alpha) = 0$ otherwise, provided that in 7.10 $\mathfrak{F}$ also satisfies that $o^{\mathfrak{F}}(\alpha) = 0$ whenever $cf(\alpha) \geq \delta$. The proof of 7.9 is unchanged; in the proof of 7.10 we need to consider the case where case 7.8 (ii) holds: i.e., $\mathfrak{F}_\nu$ is not maximal at a_ν because $a_\nu = i_\nu(a)$ and $\mathfrak{F}_M$ is not maximal at a. In this case, $cf(a) \geq \delta$ so if $a' = cf(a)$ in $K(\mathfrak{F}_M)$ then $cf(a') = cf(a) \geq \delta$ and $o^{\mathfrak{F}_M}(a') = 0$. Thus $i_\nu'' a$ is cofinal in a_ν, so $cf(a_\nu) = cf(a) \geq \delta$ and $o^{\mathfrak{F}}(a_\nu) = 0$ as well, contradicting the choice of δ.

This completes our survey of the basic properties of $K(\mathfrak{F})$. We end with illustrations of the use of the theory in finding models with large cardinals.

7.11 Theorem: Any of the following imply that there is an inner model of $\exists\kappa\ o(\kappa) = \kappa^{++}$:

(i) κ and κ^+ are both weakly compact

(ii) κ is κ^+- strongly compact

(iii) κ is measurable and $2^\kappa > \kappa^+$

(iv) κ is measurable and $\kappa^+ > \kappa^{+(K(\mathfrak{F}_M))}$

(v) there is a κ- complete ultrafilter U on κ such that $i^u(\kappa) = \kappa^+$ (in particular, AD holds).

(vi) every κ- complete filter can be extended to a κ-complete ultrafilter

(vii) there is a κ^+- saturated ideal on a successor cardinal κ.

Note that (i) and (v) imply the failure of the axiom of choice. We will prove 7.11 from more basic lemmas.

7.12 Lemma: If there is no model of $\exists\kappa(o(\kappa) = \kappa^{++})$ then there is no elementary embedding $i: K(\mathfrak{F}_M) \to K(\mathfrak{F}')$ such that $i(\kappa) > \kappa$ and $\mathfrak{F}' \restriction \kappa+1 = \mathfrak{F}_M \restriction \kappa+1$.

Proof: If there is such an embedding then, as in the proof of Lemma 6.4 we can show that $U = \{x \subset \kappa: \kappa \in i(x)\}$ is a $K(\mathfrak{F}_M \restriction \kappa+1)$-ultrafilter, contradicting the maximality of $\mathfrak{F}_M$ at κ. □ 7.12

Note that all of $\mathfrak{F}_M$ is used to ensure that U is absolutely well founded. If U is known to be countably complete then $\mathfrak{F}_M \restriction \kappa+1$ is all that is needed. This fact is used in the next proof.

Proof of 7.11 (i): Suppose κ and κ^+ are both weakly compact. Since κ^+ has no special Aronszajn trees, Specker's construction [Sp] implies that if $\lambda < \kappa^+$ and M is a model of (ZF+AC) in which $2^\lambda = \lambda$ then $\lambda^{+(M)} < \kappa^+$. In particular, if $\delta = \kappa^{++}$ in $K(\mathfrak{F}_M)$ then $\delta < \kappa^+$. Since $o^{\mathfrak{F}_M}(\kappa) \leq \delta$, $P(\kappa) \cap K(\mathfrak{F}_M)$ and $\mathfrak{F}_M \restriction \kappa+1$ can be coded by a subset A of κ. Again, $\kappa^+ > \kappa^{+(L(A))}$ so by the weak compactness of κ there is a κ

complete ultrafilter U on $P(\kappa) \cap L(A)$. Thus there is an elementary embedding $i: L(A) \to L(A')$. Now if $\mathcal{F}$ is $\mathcal{F}_M$ as defined in $L(A)$ then $\mathcal{F} \restriction \kappa+1 = \mathcal{F}_M \restriction \kappa+1$ and $j \restriction K(\mathcal{F}_M \restriction \kappa+1): K(\mathcal{F}_M \restriction \kappa+1) \to K(\mathcal{F}')$. Now $\mathcal{F}'$ is $\mathcal{F}_M \restriction i(\kappa)+1$ as defined in $L(A')$. But $A' \cap \kappa = A$, so $\mathcal{F}_M \restriction \kappa+1 \in L(A')$ and hence $\mathcal{F}' \restriction \kappa+1 = \mathcal{F}_M \restriction \kappa+1$. Since U is countably complete, Lemma 7.11 and the remark following it imply that there is a model of $\exists \kappa\ o(\kappa) = \kappa^{++}$.

$\square$ 7.1 (i)

The other clauses of Theorem 7.1 all follow from the following lemma:

<u>7.13 Lemma</u>: Suppose there is no model of $\exists \kappa (o(\kappa) = \kappa^{++})$. Then for each ordinal κ there is a δ such that $\delta < \kappa^{++}$ in $K(\mathcal{F}_M)$ and $i(\kappa) \leq \delta$ for all elementary embeddings $i: V \to N$ such that ${}^{\omega}N \subset N$.

<u>Proof</u>: Let i be any such embedding. We can assume $N = \{i(f)(w): w \in {}^{\omega}i(\kappa)\}$. Let $\mathcal{F}$ be the sequence which is maximal at ordinals α with $cf(\alpha) \leq i_{\omega}(\kappa)$ and has $o^{\mathcal{F}}(\alpha) = 0$ elswhere. Now since $i(i_{\omega}(\kappa)) = i_{\omega}(\kappa)$, $\mathcal{F}' = i(\mathcal{F})$ also has $o^{\mathcal{F}'}(\alpha) = 0$ when $cf(\alpha) > i_{\omega}(\kappa)$. Thus by Theorem 7.10and the remark following it there is an iterated ultrapower $j: K(\mathcal{F}) \to K(\mathcal{F}')$. Both $\{\nu: j(\nu) = \nu\}$ and $\{\nu: i(\nu) = \nu\}$ are λ-closed for $\lambda > j_{\omega}(\kappa)$, so both sets are thick and their intersection $\Gamma = \{\nu: i(\nu) = j(\nu) = \nu\}$ is also thick. Now let $\rho: K(\mathcal{F}'') \cong \mathcal{H}(\Gamma) \prec K(\mathcal{F})$ be the transitive collapse of the skolem hull of Γ. Then $\mathcal{F}''$ is full and there is an iterated ultrapower $k: K(\mathcal{F}_M) \to K(\mathcal{F}'')$. Then $\rho k: K(\mathcal{F}_M) \to K(\mathcal{F})$ and since $\mathcal{F} \restriction \kappa+1 = \mathcal{F}_M \restriction \kappa+1$ Lemma 7.10 implies $\rho k(\kappa) = \kappa$. In particular, $\kappa = \rho(\kappa)$ so $\kappa \in \text{range}(\rho) = \mathcal{H}(\Gamma)$ and $i(\kappa) = j(\kappa)$.

We now have an iterated ultrapower $j: K(\mathcal{F}_M) \to K(\mathcal{F})$ such that $j(\kappa) = i(\kappa)$. In addition we know that $\mathcal{F}$ is maximal in N at all ordinals less then

$i_\omega(\kappa) > i(\kappa) = j(\kappa)$. Hence if j is the direct limit of maps $j_{\nu,\nu+1}: K(\mathfrak{F}_\nu) \to \mathrm{ult}(K(\mathfrak{F}_\nu), \mathfrak{F}_\nu(a_\nu,b_\nu))$ where $\mathfrak{F}_o = \mathfrak{F}_M$, then for all ν (with $a_\nu < j(\kappa)$, which are the only ones relevant) we must have $\mathfrak{F}_\nu(a_\nu,b_\nu) \notin N$. We will first use the fact that ${}^\omega N \subset N$ to show that this implies $j(\kappa) < \kappa^{++}$. Suppose not, so $a_\nu \leq j(\kappa)$ for all $\nu < \kappa^{++}$. Since by assumption $o^{\mathfrak{F}_M}(\kappa) < \kappa^{++}$, we can find a stationary subset Γ of κ^{++} such that each ordinal in Γ has cofinality ω and if $\nu, \nu' \in \Gamma$ and $\nu < \nu'$ then $j_{\nu\nu'}(a_\nu,b_\nu) = (a_{\nu'},b_{\nu'})$. Let $\nu \in \Gamma$ be a limit point of Γ and let $(\nu_n : n \in \omega)$ be a sequence of members of Γ cofinal in ν. Then $\mathfrak{F}_\nu(a_\nu,b_\nu) = \{x \subset a_\nu : \exists m \forall n > m \; a_{\nu_n} \in x\}$. Since ${}^\omega N \subset N$, $(a_{\nu_n} : n \in \omega) \in N$ and hence $\mathfrak{F}_\nu(a_\nu,b_\nu) \in N$, which is impossible. The contradiction shows that $j(\kappa) < \kappa^{++}$.

To find a bound $\delta < \kappa^{++}$ in $K(\mathfrak{F}_M)$ we will have to refine this argument. Suppose j is an iterated ultrapower as above, obtained by taking ultrapowers by $\mathfrak{F}(a_\nu,b_\nu)$. We can always reorder the ultrapowers if necessary to ensure that $a_{\nu'} < a_\nu$ if $\nu' < \nu$. In order to make the argument clearer we will assume j has this property. We will call ν <u>good</u> if either ν is a successor ordinal, $\mathrm{cf}(\nu) > \omega$, or else ν is not a limit of ordinals $\nu' < \nu$ such that $i_{\nu'\nu}(a_{\nu'}) = a_\nu$ and $i_{\nu'\nu}(b_{\nu'}) \geq b_\nu$. We call j good if every ν is good in j. Then any j with the given properties must be good: otherwise there is a ν and a sequence $(\nu_n : n \in \omega)$ cofinal in ν such that $i_{\nu_n,\nu}(a_{\nu_n}) = a_\nu$ and $i_{\nu_n,\nu}(b_{\nu_n}) \geq b_\nu$. By the argument above we cannot have $i_{\nu_n,\nu}(b_{\nu_n}) = b_\nu$ for infinitely many n, so $b_{\nu_n} > b_\nu$ for sufficiently large n. Then $\mathfrak{F}_\nu(a_\nu,b_\nu) = \{x \subset a_\nu : \exists m \forall n > m \; x \cap a_{\nu_n} \in \mathfrak{F}_\nu(a_{\nu_n}, \lambda_n)\}$ where $\lambda_n = C(a_\nu, b_\nu, j_{\nu_n,\nu}(b_{\nu_n}))(a_{\nu_n})$ and again $\mathfrak{F}_\nu(a_\nu,b_\nu) \in M$, which is impossible.

We now define a map k inside $K(\mathfrak{F}_M)$ which is good in $K(\mathfrak{F}_M)$ by recursion on ν: If we set $\mathcal{G}_o = \mathfrak{F}_M$ and if $k_\nu : K(\mathcal{G}_o) \to K(\mathcal{G}_\nu)$ has been defined then $k_{\nu,\nu+1}: K(\mathcal{G}_\nu) \to \mathrm{ult}(K(\mathcal{G}_\nu), \mathcal{G}_\nu(c_\nu, d_\nu))$ where (c_ν, d_ν) is the least pair with $c_\nu > c_{\nu'}$ for $\nu' < \nu$ that will let ν be good for k in $K(\mathcal{F}_M)$. Since the construction is entirely inside $K(\mathcal{F}_M)$, the argument above shows that $k(\kappa) < \kappa^{++}$ in $K(\mathcal{F}_M)$. We set $\delta = k(\kappa)$, and it only remains to show that if j is any good iterated ultrapower then j can be embedded into k, so $j(\kappa) \leq k(\kappa) = \delta$. We will define an increasing map $\sigma: \ell(j) \to \ell(k)$ such that j is obtained by taking only the ultrapowers in the range of σ. This will canonically define maps $\rho_\nu: K(\mathcal{F}_\nu) \to K(\mathcal{G}_{\sigma(\nu)})$ so that $(c_{\sigma(\nu)}, d_{\sigma(\nu)}) = \rho_\nu(a_\nu, b_\nu)$ and the diagrams below commute:

$$\begin{tikzcd}
K(\mathcal{G}_{\sigma(\nu)}) \arrow[r, "k_{\sigma(\nu),\sigma(\nu')}"] & K(\mathcal{G}_{\sigma(\nu')}) \\
K(\mathcal{F}_\nu) \arrow[u, "\rho_\nu"] \arrow[r, "j_{\nu,\nu'}"] & K(\mathcal{F}_{\nu'}) \arrow[u, "\rho_{\nu'}"']
\end{tikzcd} .$$

Suppose $\sigma \restriction \nu$ has been defined and $\eta_\nu = \sup\{\sigma(\nu') + 1 : \nu' < \nu\}$ is less than the length of k. Then there is a map $\bar{\rho}_\nu: K(\mathcal{F}_\nu) \to K(\mathcal{G}_{\eta_\nu})$ and it is easy to see that there is $\sigma \geq \eta$ such that $(c_\nu, d_\nu) = j_{\eta\sigma}\bar{\rho}_\nu(a_\nu, b_\nu)$. We set $\sigma(\nu) = \sigma$, so $\rho_\nu = j_{\eta\sigma}\bar{\rho}_\nu$. Thus $\sigma(\nu)$ can be defined for all $\nu < \ell(j)$ so long as η_ν is less than $\ell(k)$. Suppose $\nu < \ell(j)$. If $\mathrm{cf}(\nu) > \omega$ then $\mathrm{cf}(\eta_\nu) = \mathrm{cf}(\nu) > \omega$ in the real world. Then η_ν certainly has cofinality greater than ω in $K(\mathcal{F}_M)$ so $\eta_\nu \neq \ell(k)$. If $\nu = \nu' + 1$ then $\eta_\nu = \sigma(\nu') + 1$ is also a successor, so again $\eta_\nu \neq \ell(k)$. Hence we can assume $\mathrm{cf}(\nu) = \omega$ and since j is good and ν is a limit ordinal there is $\nu_0 < \nu$ such that

(1) there is (a,b) such that $j_{\nu_o\nu}(a,b) = (a_\nu, b_\nu)$ and for all ν_1 with $\nu_o < \nu_1 < \nu$ we have $j_{\nu_1\nu}(a_{\nu_1}, b_{\nu_1}) < (a_\nu, b_\nu)$.

Hence $\eta_\nu < \ell(k)$ will follow from the statement that for all $\nu < \ell(k)$, if $\nu_o < \nu$ satisfies (1) then η_ν is not larger than the least $\eta > \sigma(\nu_o)$ such that $(a_\eta, b_\eta) = k_{\eta_o\eta}\,\rho_\nu(a,b)$ where $\eta_o = \sigma(\nu_o)$. But this statement can be proved by an easy induction on ν_1. $\square$ 7.13

Proof of 7.11: We have already proved (i). The hypotheses of (ii), (iii) and (iv) immediately imply that there is a map $i: V \to M$ with ${}^\omega M \subset M$ and $i(\kappa) > \kappa^{++}$ in $K(\mathfrak{F}_M)$ and hence imply the existence of a model of $\exists\kappa\ o(\kappa) = \kappa^{++}$ by Lemma 7.13. Suppose U is a κ complete ultrafilter on κ and $i^U(\kappa) = \kappa^+$. Since $K(\mathfrak{F}_M)$ satisfies the axiom of choice, $i^u(\kappa) > \kappa^+$ in $K(\mathfrak{F}_M)$ so $\kappa^+ = i^U(\kappa) \geq \kappa^{++(K(\mathfrak{F}_M))}$. Thus Lemma 7.13 implies (v) as well. We follow Kunen in proving (vi): By Lemma 3 of [K 71], if every κ complete filter over κ can be extended to an ultrafilter then for each $\delta < (2^\kappa)^+$ there is a ultrafilter U such that $i^U(\kappa) > \delta$. Thus (vi) follows from Lemma 7.13.

For (vii) we need a slight extension of 7.13. Let I be a κ^+ saturated ideal on κ. We can assume that I is normal [So]. Let P_I be the notion of forcing with conditions $x \subset \kappa$ such that $x \notin I$ and with $x < y$ if $x - y \in I$, and let U be P_I-generic over V, the universe of sets. Then U is a ultrafilter on $P(\kappa) \cap V$, κ is the first ordinal moved by $i^U: V \to M = V^\kappa/U$, and M is well founded. Since κ is a successor, say $\kappa = \lambda^+$ we have $i^U(\kappa) \leq \lambda^{+(V(U))}$. Since I is κ^+ saturated κ^+ is a cardinal in $V(U)$ so $i^U(\kappa) \leq \kappa^+$. But $i^U(\kappa) \geq \kappa^+$ so $i^U(\kappa) = \kappa^+$. Thus $i^U(\kappa) \geq \kappa^{++(K(\mathfrak{F}_M))}$. Now i is in $V(U)$ rather than in V, but the proof of Lemma 7.12 will go through if we verify that M is closed under countable sequences in $V(U)$ as well as in U and that the maximal sequence $\mathcal{F}_M$ as defined in $V(U)$ is the same as in V. Now if $\kappa \Vdash ((x_n : n \in \omega)$ is a sequence of members of $M)$ then (see [So]:) there is a sequence $(f_n : n \in \omega)$ in V such that $\kappa \Vdash \forall n\, x_n = [f_n]$. Then $(x_n : n \in \omega) = [\lambda\nu(f_n(\nu) : \nu < \kappa)] \in M$, so ${}^{\omega}M \subset M$ in $V(U)$. The other question is interesting enough to isolate as a separate lemma, which concludes the proof of Theorem 7.11.

<u>7.14 Lemma</u>: If $V(G)$ is a set generic extension of V then (i) for all $\mathcal{F}$ in V, $K(\mathcal{F})^{(V(G))} = K(\mathcal{F})^{(V)}$ and (ii) $\mathcal{F}_M$ is still a full, maximal sequence in $V(G)$.

<u>Proof</u>: Let G be P-generic over V and suppose $|P| = \delta$. Then $\nu^{+(V)} = \nu^{+(V(G))}$ for $\nu > \delta$. It follows that $\mathcal{F}_M$ is still full in $V(G)$, since if $M_\nu = K(\mathcal{F}_M \restriction \nu)$ as defined in V and ν is a singular cardinal then $\nu^+(M_\nu) = \nu^{+(V)} = \nu^{+(V(G))}$. Any new $\mathcal{F}_M \restriction \nu$ mice would collapse ν^+, so there are no $\mathcal{F}_M \restriction \nu$ mice for any ν and hence no new $\mathcal{F}$-mice for any $\mathcal{F}$. Now Lemma 7.4 has the hypothesis that there is no model of $\exists\kappa\ o(\kappa) = \kappa^{++}$, and it is not immediately obvious that this is true in $V(G)$. However the proof of 7.4 only used the existence of a full sequence, and we do know that $\mathcal{F}_M$ is still full. Hence if $\mathcal{F}_M$ is not maximal in the extension $V(G)$ then at least the $K(\mathcal{F}_M \restriction \alpha)$-ultrafilter U is unique. But it is still unique in $V(G \times G')$ if $G \times G'$ is $P \times P$-generic over V. If $U = \tau^G$ in $V(G)$ then let $U' = \tau^{G'}$ in $V(G')$. Then U and U' are $K(\mathcal{F}_M \restriction \alpha)$-ultrafilters in $V(G \times G')$, so $U = U'$. Then there cannot be a set $x \in K(\mathcal{F}_M \restriction \alpha)$ and conditions p, p' such that $p \Vdash x \in U$ and $p' \Vdash x \notin U$. Thus $U = \{x : \exists p\ p \Vdash x \in U\}$ is in the ground model V, contradicting the fact that $\mathcal{F}_M$ is maximal there. $\square$ 7.14, 7.11

*

*   The following goes at the end of section 7 of
*   "The Core Model for Sequences of Measures, II"
*
*   I have frequently refered to this paper for
*   lemma 16 below but, as Steel pointed out to me, It
*   wasn't actually in the paper.
*
*   July 1985
*
*   W. Mitchell
*

We conclude with one further lemma which, while it is not needed for the major results here, has proved extremely useful in later work. The proof is made considerably significantly simpler by using the following lemma, due to Dodd and Jensen [D lemma 8.19] which was refered to in part I of this paper:

<u>7.15 Lemma</u>: Suppose that M is a well founded model of set theory, $i\colon M \longrightarrow N$ is an iterated ultrapower, and $\sigma\colon M \longrightarrow N$ is another Σ_0 elementary embedding. Then $i(\alpha) \leq \sigma(\alpha)$ for all ordinals $\alpha \in M$.

<u>7.16 Lemma</u>: If there is no model of $\exists\kappa\ o(\kappa) = \kappa^{++}$ then every elemenatary embedding $i\colon K(\mathcal{F}_M) \longrightarrow N$ into a well founded model N is an iterated ultrapower by measures in $\mathcal{F}_M$.

<u>Proof</u>: We will prove the lemma assuming that i is set based; that is, there is a δ such that $N = \{i(f)(\xi)\colon f \in K(\mathcal{F}_M)$ and $\xi \in \delta\}$. The complete lemma follows from this, for if i is arbitrary then the maps i_δ,

$$i\colon K(\mathcal{F}_M) \xrightarrow{i_\delta} N_\delta \cong \{i(f)(\gamma)\colon f \in K(\mathcal{F}_M) \text{ and } \gamma < \delta\} \prec N,$$

are all set based and hence iterated ultrapowers. An initial segment of the iterations of i_δ will map an initial segment of $\mathcal{F}_M$ onto $i_\delta(\mathcal{F}_M) \restriction \delta = i(\mathcal{F}_M) \restriction \delta$. As δ runs through the ordinals these initial segments of the iterations i_δ will fit together to yield i.

Since i is set based it is easy to see that $N = K(i(\mathcal{F}_M))$, that $i(\mathcal{F}_M)$ is strong and full. Thus by lemma 7.10 there is an iterated ultrapower $j\colon K(\mathcal{F}_M) = K(i(\mathcal{F}_M))$, and we only need to show that $i = j$. Since i is set based, $\Gamma = \{\gamma\colon i(\gamma) = \gamma\}$ is thick, and lemma 7.15 implies that $\gamma < j(\gamma) \le i(\gamma) = \gamma$ for γ in Γ. Let

$$i'\colon N' \cong \{x\colon i(\gamma) = j(\gamma)\} \prec K(\mathcal{F}_M).$$

Then N' is full, and so there is an iterated ultrapower $j'\colon K(\mathcal{F}_M) \longrightarrow N'$. Then $j'i'\colon K(\mathcal{F}_M) \longrightarrow K(\mathcal{F}_M)$, and so must be the identity by the maximality of $\mathcal{F}_M$. Thus i' is the identity, which implies that $i = j$.

∎ 7.00

[D] A. J. Dodd, *The Core Model*, L.M.S. Lecture Notes series no. 61 (Cambridge University Press, 1982).

References

[B] E.L. Bull, Jr., Successive Large Cardinals, Ann. Math Log 15(1978) 161-191.

[De-J] K.I. Devlin and R.B. Jensen, Marginalia to a Theorem of Silver, In: Proc. of the 1974 Logic Conference, Kiel, Lecture Notes in Math, Springer Verlag, 1976.

[D-J] A. Dodd and R. Jensen, The Core Model, preprint (1977).

[Je] T. Jech, Set Theory, Academic Press, New York, 1978.

[J72] R. Jensen, The Fine Structure of the Constructible Hierarchy , Ann. Math. Logic 4(1972) 229-309.

[K-Ma] A. Kanamori and M. Magidor, The Evolution of Large Cardinals in Set Theory, In: Lecture Notes in Mathematics 699, Higher Set Theory, Müller and Scott eds., (99-275).

[K70] K. Kunen, Some Applications of Iterated Ultrapowers in Set Theory, Ann. Math. Logic 1(1970) 179-227.

[K71] K. Kunen, On the GCH at Measurable Cardinals, Logic Colloq. '69 (Proc. Summer School and Colloq, Manchester, 1969) North Holland, Amsterdam, 1971, 107-110.

[K78] K. Kunen, Saturated Ideals, Jour. Sym. Logic 43(1978) 65-76.

[Mag77] M. Magidor, On the Singular Cardinals Problem II, Ann. of Math. 106 1977, 517-547.

[Mag78] M. Magidor, Changing Cofinality of Cardinals, Fund. Math. 99 no. 1 (1978), 61-71.

[Ma-Mi] D.A. Martin and W. Mitchell, On the Ultrafilter of Closed, Unbounded Sets, Jour. Sym. Logic 44(1979) 265-268.

[M72] W. Mitchell, Aronszajn Trees and the Independence of The Transfer Property, Ann. Math. Logic 5(1972), 21-46.

[M74] Sets Constructible from Sequences of Ultrafilters, Jour. Sym. Logic 39(1974), 57-66.

[M79a] W. Mitchell, Hypermeasurable Cardinals, in Logic Colloq. '78, Boffa, Van Dalen and McAloon eds, North Holland, Amsterdam, 1979, pp. 303-316.

[M79b] W. Mitchell, Ramsey Cardinals and Constructibility, Jour. Sym. Log. 44(1979), 260-266.

[P] K.L. Prikry, Changing Measurable into Accessible Cardinals, Diss. Math. 68(1970), 5-52.

[R] L. Radin, Adding Closed Cofinal Sequences to Large Cardinals, unpublished thesis, U. of Cal. Berkeley, 1980.

[S74] J. Silver, On the Singular Cardinals Problem, Proc. of the International Congress of Mathematicians, Vancouver 1974, Vol. 1, 265-268.

[So-R-K] R.M. Solovay, W.N. Reinhardt and A. Kanamori, Strong Axioms of Infinity and Elementary Embeddings, Ann. Math. Logic 13(1978), 73-116.

[So] R. Solovay, Real Valued Measurable Cardinals, in Proceeding of Symp in Pure Math 13 part I: Axiomatic Set Theory, Dana S. Scott ed., AMS, 1970, 397-328.

[Sp] E. Specker, Sur un Problème de Sikorski, Colloq. Math. 2(1951), 9-12.

[W] H. Woodin,