25 Strong Sequences and Elementary Embeddings

In this section we return to the model $G(n)$, where $\mathbb{P}$ is a strong sequence. Our principal aim is to prove two basic theorems about $G(n)$:

2.1 Theorem: If $\mathbb{P}$ is strong then $G(n, \lambda) = G(n)$ for all pairs $(\lambda, \delta)$.

2.2 Theorem: If $\mathbb{P}$ is strong and $\mathbb{P}(\mathbb{P}) = \mathbb{Q}$ is an iterated elementary chain, then $\mathbb{P}$ is strong and $\mathbb{P} = G(\mathbb{P})$.

These theorems will be consequences of a more general result, Lemma 5.5. In order to state Lemma 5.5 we begin by proving which will be required in the next section and will first give some definitions.

2.3 Definition: (i) If $\mathbb{P} = \mathbb{I}$ is an elementary embedding and $\mathbb{I} \subseteq \mathbb{M}$ then $\mathbb{I}$ is 1-generated from $\mathbb{I}$ if $\mathbb{I} = G(\mathbb{I})$.

(iii) If $\mathbb{I} = \mathbb{H}$ is 1-generated from $\mathbb{I}$, $\mathbb{M} \supseteq \mathbb{H}$, and $\mathbb{M}^\omega = \mathbb{M}$ then $\mathbb{M}^{\mathbb{H}} = \mathbb{M}$ is defined as follow: If $\mathbb{I} \subseteq \mathbb{M}^\omega$ and $a \in \mathbb{M}^\omega$ then $\mathbb{H} = G(\mathbb{I})$, $a \subseteq G(\mathbb{I})$, and $a \subseteq G(\mathbb{I})$. If $\mathbb{M}^{\mathbb{H}} = G(\mathbb{I})$, then $\mathbb{I} = G(\mathbb{I})$. Then $\mathbb{H}$ is the class of equivalence classes $[\mathbb{H}]_G$. If $\mathbb{H}$ is well-founded under $\subseteq$ then we will identify $\mathbb{H}$ with its transitive collapse.

If $\mathbb{P} = \mathbb{I}$ is 1-generated from $\mathbb{I}$ then $\mathbb{P}$ is 1-generated from $\mathbb{I}$. More generally, if $\mathbb{P} = \mathbb{I}$ is 1-generated from $\mathbb{I}$ then $\mathbb{P}$ is 1-generated from the least $\mathbb{I}$ longer that all of the indissociables generated by $\mathbb{I}$.
If \( \mathcal{M} \upharpoonright \mathcal{R} \) is an ultrametric space then \( \mathcal{M} \upharpoonright \mathcal{R} \) is simply the ultrametric space of \( \mathcal{M} \) by the same ultrametric as used for \( \mathcal{M} \). Notice
 that the condition \( \mathcal{M} \upharpoonright \mathcal{R} \cap \mathcal{N} \) implies that the ultrametrics in \( \mathcal{N} \) are still ultrametrics in \( \mathcal{M} \). In this case, \( \mathcal{M} \) still necessarily be well

\section{Proposition}

If \( \mathcal{M} \) is strong, \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} = \mathcal{M} \), and \( \mathcal{N} \) is an ultrametric

sequence where \( \mathcal{M} \) is \( \mathcal{R} \upharpoonright \mathcal{N} \), then \( \mathcal{M} \) is strong and \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is \( \mathcal{R} \upharpoonright \mathcal{N} \).

\begin{proof}

That \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is \( \mathcal{M} \) can be proved by the same way as \( \mathcal{M} \upharpoonright \mathcal{R} \) was proved. It follows that \( \mathcal{M} = \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) \( \mathcal{M} \) is an ultrametric sequence in \( \mathcal{R} \upharpoonright \mathcal{N} \), and hence \( \mathcal{M} \) is strong.

\end{proof}

\section{Lemma}

Suppose \( \mathcal{M} \) is strong, \( \mathcal{M} \) is a set, \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is an elementary

sequence with \( \mathcal{M} \) to \( \mathcal{R} \) generated from \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) and for all strong

sequences \( \mathcal{R} \), such that \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \), if \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is the extension

of \( \mathcal{M} \) to \( \mathcal{R} \) then \( \mathcal{M} \) is well-founded. Then for all \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \)

every \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) of \( \mathcal{M} \) must be \( \mathcal{N} \).

\section{Lemma}

We have already proved the main result. Before proving theorems \( \mathcal{M} \) and \( \mathcal{R} \),

\begin{proof}

Theorem \( \mathcal{M} \) and \( \mathcal{R} \). If \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is a set then \( \mathcal{M} \) follows immediately

from \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) by taking \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) and \( \mathcal{N} \) as \( \mathcal{M} \). To prove \( \mathcal{M} \) we observe that 
(2) \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) and (5) \( \mathcal{M} \upharpoonright \mathcal{N} \). From applying \( \mathcal{M} \) \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) we observe that \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is \( \mathcal{M} \).

Theorem \( \mathcal{M} \) we have to verify that the hypothesis is satisfied.

If \( \mathcal{M} \) is a strong extension of \( \mathcal{M} \) then \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is an ultrametric

sequence by \( \mathcal{M} \). Since \( \mathcal{M} \upharpoonright \mathcal{R} \upharpoonright \mathcal{N} \) is an ultrametric sequence

by \( \mathcal{M} \). Since \( \mathcal{M} \) being an ultrametric sequence by \( \mathcal{M} \).


\begin{thebibliography}


\bibitem{2} [1981] B. C. Goodrich, "The Use of Information in the Study of


\end{thebibliography}
the system argument then shows that every transitive ultrapower of $\mathcal{M}$
is well founded (see [Mitchell, 1974], Lemma 1.1), so $\mathcal{M}'$ is well founded as required.

If $\mathcal{M}$ is not a set then we can prove 3.1 and 3.2 by picking $\nu$ with $j(\nu) = \mathcal{M}$ and applying Lemma 1.1 to the transitive ultrapower $j''(\mathcal{M}) = \mathcal{M}'$ which is obtained by using the same ultrapowers as in (3) less than $\nu$ as for $\mathcal{M}$, and skipping ultrapowers by ultrafilters on ordinals larger than $\nu$.

3.1, 3.2

Proof of 3.2: Suppose that $\mathcal{M}$, $j$, $\nu$, and $\mathcal{M}'$ are as given and that $\mathcal{M}$
is a $\mu(\mathcal{M})$-measure. We will show that $\mathcal{M}' = \mathcal{M}$. For a contradiction we could replace $j$ by

$$j''(\mathcal{M}) = \nu B_{\alpha} \lambda_{\beta} (\mathcal{M}, \mathcal{N}, \mathcal{M}) = \mathcal{M}'$$

where $\nu$ is the least ordinal greater than $\nu$ such that $e^{j''(\nu)} > 0$.

Then $j''(\mathcal{M}) = j''(\nu)$, so $\nu \notin j''(\mu)$ or $\nu \in j''(\mu)$ because $\nu \notin 0$. Also we can assume that $\mathcal{M}$ is generated from $\mathcal{M}$ since otherwise we could replace $\mu$ by $j''(\mathcal{M}) = \mathcal{M}'$, where $\mathcal{M}'$ is the transitive collapse of

$$(j''(\mathcal{M}) \in \nu \Rightarrow \nu \in j''(\mathcal{M}))$$

We will recursively define a sequence $\mathcal{M}'$ such that

1. $j''(\mathcal{M}') = \mathcal{M}'$,
2. For all $f \in \mathcal{M}'$ (domain $\mathcal{M}'$), if $\nu \in \mu(\mathcal{M})$ then $\mathcal{M}'(\nu)$ is a countably complete $\mathcal{M}'''(\nu)$-ultrafilter, and
3. For all $\nu \in \mathcal{M}'(\nu)$, $\mu(\nu(\mathcal{M}'')) = \mu(\mathcal{M}')(\nu(\mathcal{M}'')) - 1$.

Note that (1) implies that $\mathcal{M}'$ is in an ultrapower $\mathcal{M}'''$ by Theorem 3.11.

It follows that if $\nu \in \mathcal{M}'(\nu)$ then $\mu(\nu(\mathcal{M}'')) = \mu(\mathcal{M}')(\nu(\mathcal{M}'')) - 1$, so for (3)
we may need to show that \( (x, y) \in \text{def}(t) \). A sufficient (and, in fact, necessary) condition for this to hold is that

\[ (\forall v) \text{ if } x' \leq v < x \text{ then } y' = \text{def}(v) \].

So we let \( y = \text{def}(x') \) and \( y' = \text{def}(x) \).

Then we use an argument using Proposition 2.11 to show that

\[ \begin{align*}
  \text{the definition of } y' \text{ must stop at some ordinal } \delta. \\
  \text{The construction will be such that } y' \text{ will only stop if } x' \in \delta; \text{ hence we can conclude that, as claimed, the \( y' \) is } \end{align*} \]

We may need to show that \( (x, y) \in \text{def}(t) \). A sufficient (and, in fact, necessary) condition for this to hold is that

\[ (\forall v) \text{ if } x' < v < x \text{ then } y' = \text{def}(v) \].

So we let \( y = \text{def}(x') \) and \( y' = \text{def}(x) \).

Then we use an argument using Proposition 2.11 to show that

\[ \begin{align*}
  \text{the definition of } y' \text{ must stop at some ordinal } \delta. \\
  \text{The construction will be such that } y' \text{ will only stop if } x' \in \delta; \text{ hence we can conclude that, as claimed, the } \end{align*} \]
obtain ball of \( \mathbb{S} \cup \mathbb{S}^1 \) on \( g_Y^1 \) is also in \( \mathbb{S}^1 \). But \( \mathbb{S}^1 \) can be ended by a subset of \( \mathbb{S} \) and \( \mathbb{S}^1 \setminus \mathbb{S}^1 \) as \( g_Y^1 \subseteq \mathbb{S}^1 \) and hence \( Y \cap \mathbb{S}^1 \setminus \mathbb{S}^1 \).

For the rest of the proof we will assume \( H \in \mathbb{Q} \). Then \( g_Y^1 \subseteq \mathbb{S}^1 \) and \( g_Y^1 \subseteq \mathbb{S}^1 \). We set \( \mathcal{S}(v) = \mathbb{S}^1 \) unless

\[ V^1 = V^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \text{ for all } v < v. \]

Here \( A \subseteq B \) is, as in the proof of part 1 of Theorem 5.1, the last point at which \( A \subseteq B \) and \( A \subseteq B \) differ. If \( v \) holds then set \( V^1 = V^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \) and \( g_Y^1 = g_Y^1 \), and define \( (v,p) = (v, \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1) \), \( \mathcal{L}^1 \subseteq \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \) for all \( v \) such that \( \mathcal{L}^1 = \mathbb{S}^1 \setminus \mathbb{S}^1 \). We set \( (v,p) = (v, \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1) \) for all \( v \) such that \( \mathcal{L}^1 = \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \) and \( g_Y^1 \) satisfies (3) and (4) below. \( g_Y^1 \) is defined to be the least ordinal such that one of these conditions fails.

(3) \( (v,p) \) is a countably complete \( \mathcal{S}(v,p) \)-subset of \( \mathcal{S}(v,p) \).

(4) If \( (v,p) \subseteq (v',p') \) and \( \mathbb{S}^1 \) is a \( \mathbb{S}^1 \)-subset of \( (v',p') \) such that \( \mathbb{S}^1 \subseteq \mathbb{S}^1 \) and \( \mathcal{S}(v,p) \subseteq \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \) then \( \mathcal{L}^1 \subseteq \mathbb{S}^1 \setminus \mathbb{S}^1 \setminus \mathbb{S}^1 \). This completes the definition of \( \mathcal{S}^1 \). We have seen that if \( \mathcal{S}^1 \notin \mathbb{Q} \) then the construction of \( \mathbb{S}^1 \) never terminates. We will complete the proof that \( \mathbb{S}^1 \notin \mathbb{Q} \) and hence the proof of Lemma 2.1 by showing that the statement \( \mathbb{S}^1 \notin \mathbb{Q} \) implies that the construction of \( \mathbb{S}^1 \) terminates. This contradiction will show that \( \mathbb{S}^1 \notin \mathbb{Q} \).
The proof will use the fact that the maps $\lambda^\prime_0$, $\lambda^\prime_1$, and $\lambda^\prime_2$ are essentially independent of $\lambda$. Suppose $\lambda^\prime \neq \lambda^\prime_0$; then $\lambda^\prime_0 \leq \lambda^\prime$ is the automorphism of $\Sigma$ that $\lambda^\prime_0 \geq \lambda^\prime$. In particular if $\lambda^\prime$ is regular in $\Sigma$ then $\lambda^\prime_0 \geq \lambda^\prime$. Let $\lambda^\prime_0$ and $\lambda^\prime_1$ be the same automorphisms of $\Sigma$ for $\lambda^\prime_0, \lambda^\prime_0, \lambda^\prime_1$. It follows that $\lambda^\prime_0 \geq \lambda^\prime_1$ and $\lambda^\prime_2$ is the automorphism of $\lambda^\prime_0$ to $\lambda^\prime_1$. Hence we can drop the superscript $\prime$ from $\lambda^\prime_0$.

We will also drop the superscript $\prime$ from $\lambda^\prime_0$ and $\lambda^\prime_1$, to see what is going on here, define $\lambda^\prime_0 \leq \lambda^\prime_1 \leq \lambda^\prime_1$. We first need to define $\lambda^\prime_0$ and $\lambda^\prime_1$, and then $\lambda^\prime_0$, $\lambda^\prime_1$, and $\lambda^\prime_1$ agree with $\lambda^\prime_0$, $\lambda^\prime_0$, and $\lambda^\prime_1$ on the parts of their domain which were used to define $\lambda^\prime_0 \leq \lambda^\prime_1$. That we can achieve $\lambda^\prime_1 \leq \lambda^\prime_1$, $\lambda^\prime_1 \leq \lambda^\prime_1$, and this is the definition which we will actually be using. We could not have used this as the primary definition, though, because it would have been circular.

Start with $\lambda = \lambda^\prime_0 \leq \lambda^\prime_1 \leq \lambda^\prime_1$. We will use Proposition 2.21 and related arguments repeated to obtain $\lambda^\prime$ to satisfy stationarity outside of $\triangle$ and $\lambda^\prime$ has certain properties, and eventually the $\lambda^\prime$ will be strong enough to conclude that $\lambda^\prime$ is superstable.

Claim (1): $\lambda^\prime$ is $\lambda^\prime$-almost regular.

Proof: Assume $\lambda^\prime_0, \lambda^\prime_1$ and $\lambda^\prime_2$, and $\lambda^\prime_0 \leq \lambda^\prime_1$.

Claim (2): $\lambda^\prime_1$ can be chosen so that $\lambda^\prime_1 \leq \lambda^\prime_1$.

Proof: $\lambda^\prime_0 \leq \lambda^\prime_1$ and $\lambda^\prime_0 \leq \lambda^\prime_1$ so we are in Proposition 2.21.
\( \mathcal{V} \) can be shrunk so that for some fixed \( \gamma \) and all \( v \in \mathcal{V} \),
\( \mathcal{L}_n(v) = \gamma \).

**Proof:** There are \( \delta \in \mathcal{V} \) and \( v \leq \delta \) so that \( \lambda_0(v) = \delta \). We can shrink \( \mathcal{V} \) so that \( \lambda_0(v) = \delta \) is constant. If \( \lambda_0(v) = \delta \), then \( \lambda_0(v) = \delta \), as if the claim is false then we can shrink \( \mathcal{V} \) so that \( \lambda_0(v) = \delta \) for all \( v \in \mathcal{V} \).

We can shrink \( \mathcal{V} \) further so that \( \forall \gamma \in \mathcal{V} \) is constant and hence \( \lambda_0(v) = \gamma \). In particular, \( \lambda_0(v) = \gamma \). Since \( \lambda_0(v) = \gamma \), \( \lambda_0(v) \neq \gamma \). Hence \( \lambda_0(v) \neq \gamma \) as well as there is \( \lambda_0(v) \neq \gamma \) such that \( \lambda_0(v) \neq \gamma \) and \( \lambda_0(v) \neq \gamma \). According to Theorem 2.3 we can shrink \( \mathcal{V} \) so that \( \mathcal{L}_n(v) = \mathcal{L}_n(v) = \lambda_0(v) \), and conclude that \( \mathcal{L}_n(v) \neq \mathcal{L}_n(v) \) (iff \( v \in \mathcal{V} \)).

This contradicts the choice of \( \lambda_0(v) \). \( \square \) (Claim 2)

**Claim 3:** \( \mathcal{V} \) can be shrunk so that \( \lambda_0(v) = \gamma \).

**Proof:** We will first show that there are fixed ordinals \( \gamma \) and \( \delta \) such that \( \mathcal{V} \) can be shrunk so that for \( v \in \mathcal{V} \) there is \( \lambda_0(v) = \delta \) so that \( \lambda_0(v) \neq \delta \). Pick \( \gamma \) by Claim 2 so that \( \lambda_0(v) = \gamma \) for all \( v \in \mathcal{V} \). We can pick \( \delta \) so that \( \lambda_0(v) = \delta \) for all \( v \in \mathcal{V} \) and we can shrink \( \mathcal{V} \) so that \( \lambda_0(v) = \delta \) for \( v \in \mathcal{V} \).

Now every ordinal in \( \mathcal{V} \) has the form \( \lambda_0(f)(a) \) where \( f \in \mathcal{F}(\mathcal{V}) \) and \( a \) is a finite subset of \( S \cup \mathcal{L}_n(v) \). In particular, for each
In $\mathbb{T}$ we can choose $\xi$ by an ordinal and hence find $\xi \in (\mathbb{P})$ and $\xi \in K$ such that

$$\mathbb{I} \mathbb{P}(\xi) \neq \emptyset$$

Since $\xi \in K$ for all $\xi$, we can obtain $\xi$ so that $\xi \in K$ is constant. Then $\xi \in K$ for all $\xi$, as $\mathbb{I} \mathbb{P}(\xi) \neq \emptyset$ for all $\xi$, and $\xi \in K$ are as required.

Now define $\xi' = \mathbb{I} \mathbb{P}(\xi) \subseteq \mathbb{P}(\xi)$ and $\xi'' = \mathbb{I} \mathbb{P}(\xi) \subseteq \mathbb{I} \mathbb{P}(\xi)$. We will show that $\xi' \subseteq \xi''$ can be achieved so that $\xi' \subseteq \xi''$. We have $\xi' \subseteq \xi''$ and $\xi''$ satisfies the conditions $\mathbb{I}(\xi'') = \mathbb{I}(\xi')$. Hence $\xi''$ is to be $\mathbb{I} \mathbb{P}(\xi)$, hence $\xi''$ is $\xi'$. But $\xi''$ is to be $\mathbb{I} \mathbb{P}(\xi)$, hence $\xi'' = \xi'$. It follows that $\xi' = \xi''$.

Now take $\xi' = \xi''$ in $\mathbb{T}$ and let $\xi'$ be the least ordinal such that

$$\mathbb{I} \mathbb{P}(\xi') \neq \emptyset$$

Clearly $\mathbb{I}(\xi') \subseteq \mathbb{I}(\xi'')$, since $\mathbb{I}(\xi') = \mathbb{I}(\xi'')$ and $\mathbb{I}(\xi') \subseteq \mathbb{I}(\xi'')$. Hence $\xi' = \xi''$. But $\xi' \subseteq \xi''$, and $\xi' \subseteq \xi''$. Hence $\xi' = \xi''$. Hence $\xi' = \xi''$.

\[\mathbb{I}(\xi') = \mathbb{I}(\xi'') \]

Claim 3

We can show that if $\xi \in \mathbb{T}$, then $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$, for all $\xi \in \mathbb{T}$. Hence $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$.

Claim 4

We can show that if $\xi \in \mathbb{T}$, then $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$, for all $\xi \in \mathbb{T}$. Hence $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$.

\[\mathbb{I}(\xi) = \mathbb{I}(\xi) \]

Claim 5

We can show that if $\xi \in \mathbb{T}$, then $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$, for all $\xi \in \mathbb{T}$. Hence $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$. Set $\mathbb{I}(\xi) = \mathbb{I}(\xi)$. This proves Claim 5.

Claim 6

We can show that if $\xi \in \mathbb{T}$, then $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$, for all $\xi \in \mathbb{T}$. Hence $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$. Set $\mathbb{I}(\xi) = \mathbb{I}(\xi)$. This proves Claim 6.

Claim 7

We can show that if $\xi \in \mathbb{T}$, then $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$, for all $\xi \in \mathbb{T}$. Hence $\mathbb{I}(\xi) \subseteq \mathbb{I}(\xi)$. Set $\mathbb{I}(\xi) = \mathbb{I}(\xi)$. This proves Claim 7.
Theorem: Otherwise let $e_{n}$ be a sequence of sets in $W_{2}$ such that
$$\forall n \in \mathbb{N}, e_{n} \subseteq W_{2}.$$ Then, as with $\xi$ in Claim 6, $l_{n}(e_{n}) \subseteq W_{2}$ for all
$$n \in \mathbb{N}.$$ and we can define $T = \{ l_{n}(e_{n}) \mid n \in \mathbb{N} \}$.

Claim 4: $T$ can be shrunk so that $W_{2}$ is normal.

Proof: Otherwise let $\xi_{n}$ be such that $\forall n, \xi_{n}(e_{n}) \neq W_{2}$.

Claim 5: $T$ can be shrunk so that $W_{2}$ is a countably complete $\theta$-Lindelöf.

Proof: After Claim 4 and 5 we only need to prove otherwise. We show first that if $\Phi$ is a function such that $\forall n, \Phi(n) = \mathcal{P}(e_{n}) \subseteq W_{2}$ then there is $\nu \in \mathcal{P}^{\infty}(\eta)$ such that

$$(7) \quad \gamma \in l_{n}(e_{n}) \Rightarrow \nu \in l_{n}(e_{n}(\nu)) \in \nu.$$}

Observe that the function $\nu$ is compact in $W_{2}$, using $\xi_{n}(\nu_{1})$, otherwise pick $\xi_{n}$ for each $n$ so that (7) fails. Then for some $n \in \mathbb{N}$, $\nu \in l_{n}(e_{n})$

we have $l_{n}(e_{n}(\nu)) \subseteq l_{n}(e_{n}(\nu_{1})) \subseteq \xi_{n}(\nu_{1})$, as in Claim 5, shrunk.
\( \Gamma \subset E \) so that \( \lambda_1(<v,p>) \) is \( \cong_p \) and \( \lambda_2(<v,c>) \cong_p \lambda_1(<v,p>) \). Then

\( \lambda_1(<v,p>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,p>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,c>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,c>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,c>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,c>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,c>) = \lambda_2(<v,c>) \). Let \( \lambda_1(<v,c>) = \lambda_2(<v,c>) \).

It follows that if \( \alpha \in \langle \lambda_1(<v,c>), \lambda_2(<v,c>) \rangle \), then \( \lambda_1(<v,c>) : \lambda_2(<v,c>) \) such that \( [\alpha] \approx \langle \lambda_1(<v,c>), \lambda_2(<v,c>) \rangle \). To complete the proof of completeness we have to show that \( \Gamma \) can be shrunk so that for each \( \alpha \in \Gamma \) and \( \gamma \in \mathcal{P}(\alpha) \) there is \( \beta \in \mathcal{P}(\alpha) \) such that

\[
\tag{10} \quad [\gamma] \prec [\beta] \prec [\alpha] \]
Claim 6.

\textbf{Claim 6.} \textit{T} can be shrunk so that (ii) holds for all \( i \in T \).

\textbf{Proof.} We observe that \( T \) is \( \mathcal{F} \) and that (ii) holds for all \( i \in T \), and also for \( x \notin T \) if \( x \in v \) and \( x \notin \mathcal{F} \), witnessing the failure of (iv). This shows that \( T \) is not a stationary class. By the technique of Theorem 3.5 and theorem \( \mathcal{F} \) we can construct a \( \mathcal{F} \)-chain \( \langle \alpha \rangle \) such that

\[
\exists \alpha^+ = \alpha' \star \bar{\alpha}
\]

and \( R \bar{\alpha} = \beta \star \gamma + 1 \) and either \( \sigma^\gamma(v) = \beta \) or \( \beta, \gamma \notin R \bar{\alpha} \). We can find a \( T \) so that \( \mathcal{F} \notin \mathcal{F} \) and hence \( \mathcal{F} \notin T \) as \( \mathcal{F} \), \( R \bar{\alpha} \), \( R^+ \) are not \( \mathcal{F} \)-related. From \( \mathcal{F} \notin \mathcal{F} \), \( R \bar{\alpha} \), \( R^+ \) are not \( \mathcal{F} \)-related, and hence \( \mathcal{F} \notin \mathcal{F} \). Then \( \mathcal{F} \notin \mathcal{F} \) as \( \mathcal{F} \notin \mathcal{F} \) and \( \mathcal{F} \notin \mathcal{F} \). Thus \( \mathcal{F} \notin \mathcal{F} \) as \( \mathcal{F} \notin \mathcal{F} \) and \( \mathcal{F} \notin \mathcal{F} \). Then \( \mathcal{F} \notin \mathcal{F} \) as \( \mathcal{F} \notin \mathcal{F} \) and hence \( \mathcal{F} \notin \mathcal{F} \), contrary to the choice of \( T \).

\( \square \) Claim 6.

We have now shrunk \( T \) to a stationary class such that for all \( v \in T \) (i) holds and (ii) and (iv) hold for \( \alpha \star \bar{\alpha} \). But this contradicts the choice of \( \alpha = \beta \), so our assumption that the process never stops must be false.

\( \square \) Claim 6.
The covering lemma proved by Jensen for \( \mathbf{L}[\text{HOD}] \) and later by others for \( \mathbf{L} \) and \( \mathbf{L}(\mathbb{R}) \) has the clear and elegant statement that under the proper assumptions the model \( \mathbf{M} \) has a covering property: if \( x \) is any set of ordinals then there is a set \( y \in \mathbb{N} \) such that \( x \subseteq y \) and \( [y] = [x]_k \). However, this structure of indiscernibles in \( \mathbf{L}(\mathbb{R}) \) is much more complex than in \( \mathbf{L} \) and it is not clear whether the covering property can be proved for \( \mathbf{L}(\mathbb{R}) \). This problem will be discussed further in later papers; in this paper we restrict ourselves to the weak covering property.

8.1 Definition: \( \mathbf{M} \) has the weak covering property if for every sufficiently large regular strong limit cardinal \( \kappa \), \( \vec{c}(\mathbf{M}) \subseteq \kappa \).

8.2 Theorem: If there is no model of \( \mathbf{L}(\mathbb{R}) = \mathbb{R}^{\mathbf{M}} \), then there is a strong sequence \( \mathbf{F} \) having the weak covering property. If \( \mathbf{G} \) is any strong sequence with \( \mathbf{G} \subseteq \mathbf{F} \) then \( \mathbf{G} \) may be taken with \( \vec{c}(\mathbf{M}) = \mathbf{G} \).

Lemma 8.3 is proved by using the following strong version of the covering lemma:

8.3 Lemma: Suppose that there is no model of \( \mathbf{L}(\mathbb{R}) = \mathbb{R}^{\mathbf{M}} \). Then for all ordinals \( \alpha \) and sets \( A \subseteq \mathbb{P} \) there is a sequence \( B \) with \( \mathbf{B}(\mathbf{M}) \subseteq \mathbf{B} \) which is strong for \( \mathbf{L}(\mathbb{R}) \) and such that for all \( \alpha > \beta \), if \( B \) is regular in \( \mathbf{L}(\mathbb{R}) \) and \( (\mathbf{L}(\mathbb{R})(B))^{\mathbf{M}} < |B| \) then there is a \( \mathbf{L}(\mathbb{R}) \)-measurable \( \mathbf{G} \) on \( B \).

The question that \( (\mathbf{L}(\mathbb{R})(B))^{\mathbf{M}} < |B| \) can be weakened, with a little more care, to \( (\mathbf{L}(\mathbb{R})(B))^{\mathbf{M}} < |B| \). However, Lemma 8.3 will be adequate for our purpose.
Proof of 6.1 from 8.2: Let $A \subseteq \mathbb{Z}$ be a set coding the sequence $\xi$, and let $A$ be as given by 8.2. Then since $A$ is strong in $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{Q})$, $\mathcal{G}(\mathcal{Q}) \equiv \mathcal{G}(\mathcal{E}) = \mathcal{G}(\mathcal{Q})$, and hence $\mathcal{G} \equiv \mathcal{Q}$, it is strong. $\mathcal{G} \equiv \mathcal{Q}$ will be the desired sequence, because $\mathcal{G}$ is a singular strong limit cardinal generator if $A$ and $E = \mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{Q})$. Since $E(\eta) \subseteq \mathcal{Q}$, by Lemma 8.1 there is a $\mathcal{G}(\mathcal{A})$-ultrafilter on $\mathcal{Q}$, but this is impossible because $E$ is a nonmeasurable set in $\mathcal{G}(\mathcal{A})$.

Proof of 6.2: The sequence $\mathcal{G}$ is defined recursively. Let $\mathcal{G}(0) = \eta$.

If $\mathcal{G}(\mathcal{A})$ has been defined and there is a countably complete
$\mathcal{G}(\mathcal{A})$-ultrafilter then pick any such ultrafilter for $\mathcal{G}(\mathcal{A})$.
Otherwise set $\mathcal{G}(\eta) = \mathcal{G}$. By (the relativization of) Lemma 6.1, $\mathcal{G}$ is strong in $\mathcal{G}(\mathcal{A})$.

In the following we shall frequently simply ignore the set $A$. The presence of $A$ has no effect on the proof except that deriving relativizations of earlier results are used.

Let $\mathcal{G}$ be regular in $\mathcal{G}(\mathcal{E})$ and suppose $\{u \in \mathcal{E} \cap \mathcal{G}(\mathcal{A}) \cup \{\xi\} \}$.
Choose a coherent subset $\mathcal{E}$ of $\mathcal{E}$ such that $|\mathcal{E}| = \mathcal{G}(\mathcal{E})$ and let $\mathcal{E}$ be an
elementary substructure of $\mathcal{E}_\mathcal{E}$, the sets of rank less than $\mathcal{E}_\mathcal{E}$, such that

\begin{align*}
\mathcal{E} &= (\mathcal{E} \cup \mathcal{G}(\mathcal{E}))^\mathcal{E} \\
\mathcal{E} &= (\mathcal{E} \cup \mathcal{G}(\mathcal{E}))^\mathcal{E} \\
\mathcal{E} &= (\mathcal{E} \cup \mathcal{G}(\mathcal{E}))^\mathcal{E}
\end{align*}
Let $x = 0$ or $x$ for a transitive set $x$, and set $Z = v^{-1}(x)$ and $\mathcal{F} = \mathcal{V}^{x}(x)$. The proof of 6.1 breaks into two cases, depending on whether or not $\mathcal{M}(x) \cap \mathcal{F} \subseteq \mathcal{V}^{x}(x)$ for all $\mathcal{M} \in \mathcal{F}$.

Case 1: $\mathcal{M}(x) \cap \mathcal{F} \subseteq \mathcal{V}^{x}(x)$ for all $\mathcal{M} \in \mathcal{F}$. We will show that in this case there is a model of $\mathcal{W}(x) = \mathcal{V}^{x}(x)$ under the hypothesis of the case 2

Case 2: $\mathcal{M}(x) \cap \mathcal{F} \not\subseteq \mathcal{V}^{x}(x)$ for all $\mathcal{M} \in \mathcal{F}$. Now fix a set $\mathcal{M} \in \mathcal{F}$ such that $\mathcal{M}(x) \not\subseteq \mathcal{V}^{x}(x)$. Suppose $\mathcal{M}(x) \cap \mathcal{F} \subseteq \mathcal{V}^{x}(x)$. If $\mathcal{F}(x) \cap \mathcal{V}^{x}(x) \subseteq \mathcal{V}^{x}(x)$, we can extend the model $\mathcal{M}(x)$ to a model $\mathcal{M}'(x)$ satisfying $\mathcal{M}'(x) \supseteq \mathcal{M}(x)$ and $\mathcal{M}'(x) \cap \mathcal{F} \subseteq \mathcal{V}^{x}(x)$.

Proposition 5.3.1 (1) The map $\mathcal{M}(x) \to \mathcal{F}(x)$ preserves an extension $\sigma: \mathcal{M}(x) \to \mathcal{F}(x)$. If $\mathcal{F}(x) \subseteq \mathcal{V}^{x}(x)$ is well founded, then we can extend it with a transitive class $x$.

Proposition 5.3.2 Suppose $\mathcal{M} \in \mathcal{F}(x)$ is an elementary embedding there. $\mathcal{M}$ is sufficiently large substructure of $\mathcal{M}(x)$, $\mathcal{M}(x) \cap \mathcal{F} \subseteq \mathcal{V}^{x}(x)$ whenever...
to the first cardinal named by \( \kappa \), and \( \kappa(\kappa) \) is well-founded.

\[ \kappa(\kappa) = \sup \{ \kappa(\alpha) : \alpha < \kappa \} \]

where \( \kappa(\kappa) = \sup \{ \kappa(\alpha) : \alpha < \kappa \} \) is well-founded.

Proof: "Sufficiently large" will be explained by the proof. In particular

the \( \kappa \) given in Case 1 works. By modification of Theorem 5.2,

\[ (\kappa(\kappa))^\kappa = \lambda \]

for a strong sequence \( \lambda \). We first show that

\[ \kappa(\kappa)^{\kappa} = \lambda \]

This is the limit cardinal and \( \eta \) is a limit cardinal in

\[ \kappa(\kappa) \]

Then \( \eta \) is a limit cardinal in \( \kappa(\kappa) \).

Any cardinal \( \eta \) less than \( \kappa(\kappa) \)

can be represented by the pair \( (\alpha, \beta) \). The cardinal \( \kappa^\alpha \) can be

represented by the pair \( (\alpha^\beta, \beta) \) on \( \eta \) and \( \kappa \) is a limit cardinal on

\[ \kappa(\kappa) \]

\[ \kappa(\kappa)^{\kappa} = \lambda \]

i.e., \( \lambda = \kappa(\kappa) \).

Thus \( \kappa(\kappa)^{\kappa} = \lambda \).

If \( \xi \in \kappa(\kappa) \) then \( \kappa(\kappa)^{\kappa} = \lambda \)

is represented by the pair

\[ (\xi, \kappa(\kappa), \kappa(\kappa)^{\kappa}, \kappa(\kappa)^{\kappa}, \kappa(\kappa)^{\kappa}) \]

and every element of \( \kappa(\kappa) \) is represented by

the pair \( (\alpha, \beta) \).

Thus \( \kappa(\kappa)^{\kappa} = \lambda \).

Thus \( \kappa(\kappa)^{\kappa} = \lambda \).

I.e., \( \lambda = \kappa(\kappa) \).

For we have an elementary submodel \( \kappa(\kappa)(\kappa) = \kappa(\kappa) \)

with

\[ \kappa(\kappa) \]

and every \( \kappa \) is well-founded. We will complete the proof by showing that if \( \kappa(\kappa) \) is

then \( \kappa(\kappa) \) is also well-founded. Let \( \kappa(\kappa) = \kappa(\kappa) + 1 \)

and consider the commutative triangle:

\[
\begin{array}{ccc}
\kappa(\kappa) & \xrightarrow{1} & \kappa(\kappa)\\
\kappa(\kappa) & \downarrow & \kappa(\kappa) \\
\kappa(\kappa) & \xrightarrow{1} & \kappa(\kappa)
\end{array}
\]
where \( E \alpha(x) \equiv \alpha F(x) \) for any member \( E \alpha \) of the ultrapower.

The claim that \( U \) is isometric in \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \) translates straightforwardly to the claim that the first ordinal moved by \( k \) is greater than \( \mathcal{M}(\mathcal{M} + 1) \), so that \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) > k \). By assumption we have \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \leq \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(2) \), so \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) < \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(2) \). Hence \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \) is a set. This will be enough to show that the first ordinal moved by \( k \) is at least \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \) in \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \). Now \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \) is the identity, and if \( \mathcal{T} \subseteq \mathcal{M}(\mathcal{M} + 1) \) there is a subset of \( \mathcal{M} \) of order type \( \gamma \) in \( \mathcal{M}(\mathcal{M} + 1) \) and hence in \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \), so \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \) is the identity. If \( \mathcal{T} \subseteq \mathcal{M}(\mathcal{M} + 1) \) and hence in \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \), then \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \) is the identity. Otherwise \( \mathcal{T} \subseteq \mathcal{M}(\mathcal{M} + 1) \) and \( \mathcal{T} \subseteq \mathcal{M}(\mathcal{M} + 1) \), contrary to assumption. But the first ordinal moved by \( k \) is a member of \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \) and hence in \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \), so it must be at least \( \mathcal{M}(\mathcal{M} + 1) + \mathcal{T}(1) \). This proves the claim.

Proof of (2.9), continued: We now show that if \( \mathcal{T} \subseteq \mathcal{M}(\mathcal{M} + 1) \), then as defined before and \( \mathcal{M}(\mathcal{M} + 1) = (\mathcal{T} \mathcal{M})^{\mathcal{M}} \) then \( \mathcal{E}(\mathcal{T} \mathcal{M})^{\mathcal{M}} \) is well-founded. It then follows by Lemma 6.1 that the function \( f \) is \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \)-hypertame.

But \( \mathcal{T} \) is also countably complete. Otherwise let \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \) be a sequence of sets in \( \mathcal{T} \) such that \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) = \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \) and hence \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) = \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \). But this leads to a contradiction since \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) = \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \). Hence \( \mathcal{T} \) \( \mathcal{M}(\mathcal{M} + 1) \) is a countably complete \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \)-hypertame, contradicting the definition of \( \mathcal{E}(\mathcal{T} \mathcal{M})^{\mathcal{M}} \).

If \( \mathcal{E}(\mathcal{T} \mathcal{M})^{\mathcal{M}} \) is not well-founded then let \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \) be a sequence of sets in \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \) such that \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) = \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \) and hence \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) = \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \). But this leads to a contradiction since \( \mathcal{T} \mathcal{M}(\mathcal{M} + 1) = \mathcal{T} \mathcal{M}(\mathcal{M} + 1) \). Hence \( \mathcal{T} \) \( \mathcal{M}(\mathcal{M} + 1) \) is a countably complete \( \mathcal{U} \mathcal{P}(\mathcal{M} + 1) \)-hypertame, contradicting the definition of \( \mathcal{E}(\mathcal{T} \mathcal{M})^{\mathcal{M}} \).
In case 1, note that if \( \langle x, z \rangle \in \mathcal{R}(\mathcal{F}) \) then \( \langle x, z \rangle \in \mathcal{S}(\mathcal{F}) \), so \( \langle x, z \rangle \in \mathcal{R}(\mathcal{F}) \) is a decreasing sequence of ordinals, which is impossible.

**Case 2.** \((\forall x)(\exists y)[y \in \mathcal{R}(\mathcal{F}) \text{ for some } y < x]\). In this case we will either show that \( \mathcal{F} \) is singular in \( \mathcal{R}(\mathcal{F}) \) or else construct indiscernibles for \( \mathcal{R}(\mathcal{F}) \) which can be used to define a \( \mathcal{R}(\mathcal{F}) \)-ultrahomomorphism of \( \mathcal{F} \). The last alternative is excluded by the hypothesis of the lemma 6.1 on which we are relying here, so the \( \mathcal{R}(\mathcal{F}) \)-ultrahomomorphism required by the conclusion must exist. It should be remarked that this is the most interesting of the two possibilities, although this fact will not be apparent in the truncated version given here.

In case 1, larger ordinals exist than can be dealt with in \( \mathcal{R}(\mathcal{F}) \), so the only information given by the argument in that case is that the theory stays unachieved by reality. In case 2, on the other hand, all larger ordinals exist in \( \mathcal{R}(\mathcal{F}) \) and the proof gives quite a bit of information about how the universe of sets is built up from a base in \( \mathcal{R}(\mathcal{F}) \).

If the hypothesis of case 2 holds, then let \( \mathcal{F} \) be the least \( \mathcal{R}(\mathcal{F}) \)-measure such that \( \mathcal{F}(\mathcal{F}) \cap \mathcal{F} = \emptyset \). For some \( x \in \mathcal{E} \), then \( \mathcal{F}(\mathcal{F}) \), \( x \in \mathcal{E} \) and some sequence \( y \) with \( \mathcal{F}(\mathcal{F}) = \emptyset \). We will carry out, as far as possible, a few more analyses of \( \mathcal{F}(\mathcal{F}) \) as in Section 4. Since \( \mathcal{F} \) is an ultrahomomorphism sequence, there is no problem for all \( x \) such that the projection \( p_x \) is not smaller than \( x \). In the other hand there must be an \( x \) such that \( p_x \subset \mathcal{E} \), chosen by the minimality of \( \mathcal{F} \). Every subset of \( \mathcal{F}(\mathcal{F}) \), \( x \in \mathcal{F}(\mathcal{F}) \), and \( \mathcal{E} \) is an indiscernible sequence satisfying \( x \in \mathcal{F}(\mathcal{F}) \) of \( \mathcal{E} \), and \( \mathcal{F}(\mathcal{F}) \) has a non-\( \mathcal{F}(\mathcal{F}) \)-ultrahomomorphism of some \( y \in \mathcal{F}(\mathcal{F}) \). Now let \( \mathcal{F}(\mathcal{F}) > 0 \), then, since \( p_x \subset \mathcal{E} \), \( \mathcal{F}(\mathcal{F}) \) is a \( \mathcal{F}(\mathcal{F}) \)-ultrahomomorphism sequence and \( \mathcal{F}(\mathcal{F}) \) is nontrivial.
In the context parameter $p$, whether or not $\delta(p) > 0$, let $\mathcal{C}$ be the system of invariance vectors for the reduction, so that $\mathcal{C} \cong \mathcal{C}(\mathcal{D}),$

Let $\tau \leq \mathcal{F}$ be the least ordinal above $\tau$, $\mathcal{C} \cong \mathcal{C}$, segment in $\mathcal{C}$, from $\tau \pi$.}

Claim: $\mathcal{C}(\mathcal{A}) \leq \mathcal{C}(\mathcal{B})$ is reducible in $\mathcal{C}$.

Proof: By the minimality of $\mathcal{C}$, every set in $\mathcal{C}$ is in $\mathcal{C}$, and the map $\pi: \eta \to \eta$ defines a $\mathcal{C}$-elementary map $\pi: \mathcal{C} \to \mathcal{C}$ such that $\eta = \pi(\eta)$. Since $\eta$ is $\mathcal{C}$-elementary, $\mathcal{C}$ can be defined by a structure $\mathcal{C}$, such that $\mathcal{C} \cong \mathcal{C}$, and $\mathcal{C}$ has a rich and complete system of reductions above $\tau$.

But show $\mathcal{C}$ is a nontotalizing sequence (above $\tau$) in $\mathcal{C} \cong \mathcal{C}(\mathcal{B})$. The $\mathcal{C}$-theory of $\mathcal{C}$ is a member of $\mathcal{C}$, and $\mathcal{C}$ can be collapsed to give a $\mathcal{C}$-elementary $\mathcal{C}$-theories $\mathcal{C}$, where the $\mathcal{C}$-theory, and hence $\mathcal{C}$ itself, is in $\mathcal{C}(\mathcal{C})$, and hence by Theorem $3.1$, is in $\mathcal{C}(\mathcal{C})$.

Note also that $\mathcal{C}(\mathcal{C}) \subseteq \mathcal{C}(\mathcal{C})$ is bounded in $\mathcal{C}$ and let $\sigma$ be the sup of $\mathcal{C}(\mathcal{C}) \subseteq \mathcal{C}(\mathcal{C})$. Then the $\mathcal{C}$-filter of $\mathcal{C}$ is equal to $\mathcal{C}$.

Since $\mathcal{C}$ is definable in $\mathcal{C}$, it follows that the $\mathcal{C}$-filter of $\mathcal{C} \cup \mathcal{F}(\mathcal{C})$ is $\mathcal{C}$-elementary in $\mathcal{C}$, and hence the $\mathcal{C}$-filter of $\mathcal{C} \cup \mathcal{F}(\mathcal{C})$ is $\mathcal{C}$-elementary in $\mathcal{C}$, and it follows that $\mathcal{C}$ is singular, contradicting the assumption that $\mathcal{C}$ is regular in $\mathcal{C}(\mathcal{C})$.

Claim:

Now let $\pi \in \mathcal{C}(\mathcal{C}) \subseteq \mathcal{C}(\mathcal{C})$, and for $x \in \mathcal{C}(\mathcal{C})$ let

$\mathcal{C}(x) = \{x \in \mathcal{C}(\mathcal{C}) \mid \mathcal{C}(x) \in \mathcal{C}(\mathcal{C})\}$

and $\mathcal{C}(x_0) = \{x_0 \in \mathcal{C}(\mathcal{C}) \mid \mathcal{C}(x_0) \in \mathcal{C}(\mathcal{C})\}$. We claim that $\mathcal{C}(x)$ is the $\mathcal{C}(\mathcal{C})$-definition of $\mathcal{C}(x)$ described by Lemma $3.2$.}
Suppose $f \in K(\Omega)$ and $(\forall n \in \omega) \alpha_n \in \theta$. For all sufficiently large $n \in \omega$ there is an ordinal $\gamma \in \text{cof}(\alpha_n)$ such that $(\forall n \in \omega) \alpha_n \in \theta$. Let’s call this ordinal $\gamma(n)$. Then there is an increasing sequence $(\beta_n : n \in \omega)$ in $\theta$ such that $\alpha_n \leq \beta_n$.

Now since $(\forall n \in \omega) \beta_n \in \theta$, the sequence $(\gamma(n) : n \in \omega)$ is in $\Theta$ and hence so is the sequence $(\gamma(n) : n \in \omega)$ where $\gamma(n) = \gamma(n) \cup \beta_n$.

Since $\beta \in \text{cof}(\theta)$, elementary it is true in $\theta$ that there is a function $g \in \text{cof}(\theta)$ such that for each $n \in \omega$ but if $g(n)$ is the ordinal such that $(\forall n \in \omega) \gamma(n) = \gamma(n) \cup \beta_n$ then $\gamma(n) \neq \beta_n$ whenever $n \neq n_0$. Since $g \in K(\Theta)$, $g$ is in $\text{cof}(\theta)$ and hence $\beta \in \text{cof}(\theta)$. Then $g(n)$ is in the $\gamma(n)$-th ordinal of $(\gamma(n) : n \in \omega)$ and hence $g(n) = g(n) \cup \beta_n$ for some $g(n) \in \beta_n$. Some sequence $s$ and some $\beta$ function $\tau$, we use assume, by deleting an initial segment of $(\beta_n : n \in \omega)$ if necessary, that for each $n \in \omega$ we have either $\beta < \beta_n$ or for each $n \in \omega$.

Now if there are integers $x < k$ such that $\gamma(x) \neq \gamma(k)$ then $\beta(x) \neq \beta(k)$ is equivalent to $\gamma(x) \neq \gamma(k)$. Since $g(y) \leq \gamma(x)$ and $\gamma(y) = \gamma(k)$ we have $\gamma(x) \leq g(y) \leq \gamma(k)$ contrary to assumption. Similarly if there are integers $x \neq y$ and $x < k$ such that $\gamma(x) \neq \gamma(y)$ and $\gamma(x) \neq \gamma(k)$ then if $\beta(x) = \gamma(x)$ is the ordinal $y$ such that $(\forall n \in \omega) \gamma(n) = \gamma(n) \cup \beta_n$ then $\gamma(x) \neq \gamma(y)$ is derivable from $\beta(x) \neq \beta(y)$ so $\gamma(x) = \beta(x) \neq \gamma(y)$ contrary to assumption. Finally, if there are integers $x \neq y$ such that $\gamma(x) \neq \gamma(y)$ and $\gamma(x) \neq \gamma(k)$ and $\gamma(x) \neq \gamma(y)$ is derivable from $\beta(x) \neq \beta(y)$ then $(\forall n \in \omega) \gamma(n) = \gamma(n) \cup \beta_n$. If otherwise it follows that $(\forall n \in \omega) \gamma(n) = \gamma(n) \cup \beta_n$ so again $\gamma(x) = \gamma(y)$ contrary to assumption. But there must be some pair of integers $x \neq y$ such that $\beta(x) \neq \beta(y)$ and $\gamma(x) \neq \gamma(y)$ on the ordinals $\gamma(x)$ cannot all be distinct. \[\]
§7 Applications of the Weak Covering Property

In this section we will list the antecedent of sequences with the weak
covering property to prove for \( WIF \); some of the results which have proved
true for \( A(I) \) in \( [41] \) and which were extended to \( WIF \), under a more restrictive
hypothesis, in \( [43, 55] \). We will also show that those results are not always
true for \( WIF \).

All of the results of this section are proved without the axiom of choice
beyond dependent choice. The only one which has been made so far of the full
axiom of choice is in the inductive definition of the sequence \( D \), in which
we were required to choose an ultrafilter to be \( \mu_{\mathcal{F},y} \) whenever possible.
In Lemma 2.6 we will show that there is only one possible choice of \( \mu_{\mathcal{F},y} \)
and hence the axiom of choice is not required.

In Theorem 7.11 we will prove those results promised in the introduction
that various hypotheses imply the existence of models satisfying \(-\forall^\infty \).

We [43, 55] present one is made of the fact that \( T \) is a proper class
that every subset of \( L(\mathcal{F}) \) of an ordinal \( \alpha \) is definable in \( L(\mathcal{F}) \) from parameters
in \( \alpha \). This is true of \( L(\mathcal{F}) \) because the transitive collapse of the closure
ball of \( L(\mathcal{F}) \) is \( L(\mathcal{F}) \); a \( L(\mathcal{F}) \) contains all the ordinals and hence must be
\( L(\mathcal{F}) \) for some \( \beta \). In order to work in \( WIF \) it is not enough to know
that the transitive collapse of \( WIF \) contains all the ordinals; it is also necessary to know that it contains \( \alpha \) and \( \beta \). The weak covering property
will be used for this purpose.

2.1 Definition: A class \( T \) is \( \alpha \)-closed for a regular cardinal \( \alpha \) if \( T \)
contains \( \alpha \)-closed and unbounded subclasses \( \beta \) such that \( \beta \in \alpha \cap \mathcal{F} \).
for all $x \in E$. A sequence $z$ is $\rho$-full for a regular $\rho$ if there is an $\omega$-closed, unbounded class $\xi$ of ordinals such that if $x \in z$ then $x^\omega$ is $\xi$. 

A class is said to be thick, or a sequence to be full, if it is $\omega$-thick or $\omega$-full for all sufficiently large regular $\omega$.

Note that Theorem 5.1 asserts that any strong sequence has an extension which is full.

If $\Gamma$ is an intended ultrafilter of $E(P)$ then we call $i$ proper if the order type of $i(E(P))$ is at most $\omega$ or, equivalently, if no single ordinal is in $\bigcup$ many times in the ultrafilter.

**Proposition:** Suppose $\mu$ is $\rho$-full and $\Gamma$ is $\omega$-thick.

(i) any intersection of $\rho$-thick classes is $\omega$-thick.

(ii) If $i$ is a proper increased ultrafilter of $E(P)$ then $i(E(P))$ is full.

(iii) $\rho(x) = 0$ whenever $i(x) = 1$ then $i(i(x) + 1)$ is $\rho$-thick.

(iv) If $\mu$ and $\xi$ are each $\rho$-full then there is a $\tau$-full sequence

\[ x \in E(P) \quad \text{and} \quad y \in E(P) \]

\[ i(x) \neq 0 \]

(v) If $\mu$ is isomorphic to the skeleton $E(U \cdot q)$ of $\Gamma \cup \gamma$ in $E(P)$ then $\mu = E(P)$ for a $\omega$-full sequence $\mu$.

For all ordinals and sets $x \subseteq \kappa$ in $E(P)$, $\kappa$ is definable from parameters in $\Gamma \cup \kappa$.

**Proof:** Choose (i) to close and choose (ii) to close unless the iteration has length $\omega$. If it has length $\omega$ then suppose $i$ is the limit of
Let \( F = \{ x : x \leq \sqrt{2} \} \) and let \( \mathcal{F} \) be the class of ordinals \( x \) such that \( (L_x) = x \) and \( (R_x) = x \). Then \( \mathcal{F} \) consists of closed unbounded class and for any \( x \in \mathcal{F} \) we have \( x \) is \( (L_x) \) and \( x \) is \( (R_x) \). If \( x \) is an ordinal and \( x \in \mathcal{F} \), which is \( \mathcal{F} \). If \( x \) is an ordinal and \( x \in \mathcal{F} \), then \( x \) is a member of \( \mathcal{F} \) or \( x \notin \mathcal{F} \). Thus \( (L_x) = x \).

To prove class (i), define the ultrapowers \( i \) and \( j \) in measure \( \mathcal{F} \) and \( \mathcal{Q} \) as in Theorem 3.5. By the proof of Theorem 3.5 at least one of the classes \( \{ x : x \in \mathcal{F} \} \) and \( \{ x : x \notin \mathcal{F} \} \) is closed for each ordinal \( x \).

Now or least one of \( i \) and \( j \) is proper; suppose \( i \) is proper. If \( j \) is also proper we are done, so suppose \( j \) is not proper, so \( \mu(\mathcal{F}) = \mu(\mathcal{Q}) \). There is an ordinal \( \nu_0 \) such that \( \mathcal{F} = \{ x : \nu_0 < x \} \). For all \( \nu \in \mathcal{F} \), \( \nu \) is the least ordinal such \( \mu(\mathcal{F}) \) has real cardinality \( \nu \) and \( \nu \) is less than the real \( \nu \). This is impossible, since \( \mu(\mathcal{F}) = \mu(\mathcal{Q}) \) and \( \mu(\mathcal{Q}) = \nu \) is \( \nu \).

To prove class (ii), let \( \mathcal{D} = \mathcal{F} \cup \mathcal{Q} \), where \( \mathcal{D} = \mathcal{F} \cup \mathcal{Q} \), where \( \mathcal{D} = \mathcal{F} \cup \mathcal{Q} \). There is a closed class \( \mathcal{C} \) of ordinals \( x \) such that \( (L_x) = x \) and \( (R_x) = x \). Then for \( x \in \mathcal{C} \) we have \( x \notin \mathcal{C} \), so \( x \) must contain all of \( \mathcal{C} \) and \( x \). Hence \( \mathcal{D} \) is the class of \( \mathcal{C} \) and \( \mathcal{D} = \mathcal{C} \).

If \( \mathcal{F} \) and \( \mathcal{Q} \) are as in class (ii) then by (ii) we can take \( \mathcal{D} = \mathcal{F} \cup \mathcal{Q} \) and \( \mathcal{D} = \mathcal{D} \). Then by (iii) there is a proper (closed) ultrapower \( \mathcal{D} \) and \( \mathcal{D} \) is \( \mathcal{D} \). Then any subset \( x \) of \( x \) is \( x \) and hence in \( \mathcal{D} \), so \( x \) is definable from parameters in \( \mathcal{D} \).
Using 7.3(i) we can easily extend 6.2 to get:

L.3. Lemma: If $e$ is a strong sequence and $d_{e}(e) > 0$, then there is a
strong, full $\mathfrak{A}$ such that $\Pi_{e}(\mathfrak{A}) = \mathfrak{A}$, every $\text{for}(a)$ is
suitably complete, and $\lambda_{e}(a) = a$ for all $a$ with $\text{for}(a) \in \mathfrak{A} \cup \mathfrak{Q}$.

-\lambda_{e}(a) = a$ for all $a$ with $\text{for}(a) \in \mathfrak{A} \cup \mathfrak{Q}$.

L.4. Lemma: Suppose $\mathfrak{A}$ is a strong sequence, $\mathfrak{A}' \leq \mathfrak{A}$, and $\mathfrak{Q}$ and $\mathfrak{Q'}$ are
BPH-assortments on $\mathfrak{A}$. Then $\mathfrak{Q}' = \mathfrak{Q}$.

Proof: If $\mathfrak{Q} \neq \mathfrak{Q}'$, then the lemmas also apply to $\Pi_{e}(\mathfrak{A}'_{e} \mathfrak{A} e)$, and the
same of choice holds. Thus we can assume the same of choice. By Lemmas 6.4 and 6.5 there is a full sequence $\mathfrak{Q}$ such that $\lambda_{\mathfrak{Q}'}(e) = \mathfrak{Q}$ and $\lambda_{\mathfrak{Q}'}(e) = \mathfrak{Q'}$ whenever $\text{for}(e) = 1$. By Lemma 7.2 there are suited assortments

\[
\begin{align*}
\mathfrak{Q}'' = & \mathfrak{Q}''(\mathfrak{A}', \mathfrak{A}) \\
\mathfrak{Q}' = & \mathfrak{Q}'(\mathfrak{A}''(\mathfrak{A}') \mathfrak{A} e, \mathfrak{A})
\end{align*}
\]

It is easy to see, using the fact that $\lambda_{\mathfrak{Q}}(e) = e$ for all $e$ with
$\text{for}(e) = 1$, that $\mathfrak{Q}''(\mathfrak{A}', \mathfrak{A})$ is a full $\mathfrak{A}'$-clique. Thus $\lambda_{\mathfrak{Q}'}(e) = \mathfrak{Q}''(\mathfrak{A}', \mathfrak{A})$. By Proposition 7.3(i),
any subset $a$ of $a$ in $\mathfrak{Q}''(\mathfrak{A}', \mathfrak{A})$ is definable from parameters in $a \mathfrak{A}'$. It
follows that $\lambda_{\mathfrak{Q}'}(e) = \lambda_{\mathfrak{Q}''(\mathfrak{A}', \mathfrak{A})}(e)$. Set $a \subseteq \mathfrak{Q}'$ iff $a \subseteq \lambda_{\mathfrak{Q}''(\mathfrak{A}', \mathfrak{A})}(e)$, which holds
iff $a \subseteq \lambda_{\mathfrak{Q}'}(e)$, and similarly for $\mathfrak{Q}'$. Thus $\mathfrak{Q}' = \mathfrak{Q}$.

L.4. Lemma: If $\mathfrak{Q}_{1}$ and $\mathfrak{Q}_{2}$ are strong, and $\lambda_{\mathfrak{Q}'}(e) = \lambda_{\mathfrak{Q}''}(e)$ for all $e$ with
$\text{for}(e) = 1$ and $\lambda_{\mathfrak{Q}'}(e) = \lambda_{\mathfrak{Q}''}(e)$ for all $e$ then $\mathfrak{Q}_{1} = \mathfrak{Q}_{2}$.

This is not true if we look at $\mathfrak{Q}$ instead of $\mathfrak{A}$. The following
example answers a question left open in (K M).
1.5 Measure: Suppose \( \kappa \) is measurable and there are at least \( \kappa \) measurable cardinals. Then there are disjoint sequences \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of coherently complete filters such that \( \mathcal{F}_i(x) \leq 1 \) for all \( x \) and \( \mathcal{F}_i \) is an ultrafilter sequence in \( \mathcal{M}(\mathcal{F}_i) \) for \( i = 1, 2 \).

Proof: Since we are dealing with only one measure per cardinal, we will identify our measures somehow; let \( (\mathcal{F}_i)_{\alpha \in \kappa} \) be an increasing sequence of cardinals and let \( \mathcal{F}(x) \) be an ultrafilter on \( 
abla \kappa \alpha \). We can assume that \( \mathcal{M}(\kappa) \).

Claim: For all \( \alpha \in \kappa \), there is \( \aleph_1 \leq \kappa \) such that \( \alpha \leq \aleph_1 \).

Proof: Since \( \kappa = \aleph_1 \), every subset \( \kappa \) of \( \kappa \) is of \( \aleph_1 \) measures. Now we can assume \( \kappa \) is a "post measure" (see [476]), that is, a model of \( \kappa \). That the proof of theorem 7.3 does not need the use of any of the auxiliary developed in this paper.

Now we compute the length of \( \kappa \) with \( \kappa \), as in Theorem 7.3. Since we are working in \( \kappa \), there must be \( \aleph_1 \) \( \aleph_1 \)'s, or \( \aleph_1 \)'s, on \( \kappa \), and so \( \kappa \) is a "post measure" such that \( \kappa = \aleph_1 \). The proof of theorem 7.3 does not need the use of any of the auxiliary developed in this paper.

Let \( \kappa = \aleph_1 \), so we can assume \( \kappa \) of \( \aleph_1 \)'s, or \( \aleph_1 \)'s, on \( \kappa \), and so we are working in \( \kappa \). Since \( \kappa = \aleph_1 \), there are \( \aleph_1 \)'s, or \( \aleph_1 \)'s, on \( \kappa \), and so we are working in \( \kappa \). Since \( \kappa = \aleph_1 \), there are \( \aleph_1 \)'s, or \( \aleph_1 \)'s, on \( \kappa \), and so we are working in \( \kappa \). Since \( \kappa = \aleph_1 \), there are \( \aleph_1 \)'s, or \( \aleph_1 \)'s, on \( \kappa \), and so we are working in \( \kappa \).
such that for some \( \delta > 0 \), \((\gamma_0, \delta, \gamma_0)\) is a sequence of measures in \( L^p (\mathbb{R}) \) on the intervals \( (\gamma_0, \delta) \). Since \( \mu' (\mathbb{R}) \) has a finite measure \( \| \mu' (\mathbb{R}) \| \), \((\gamma_0, \delta, \gamma_0)\) is a sequence of measures in \( L^p (\mathbb{R}) \).

Next, let \( \nu \subset L^p (\mathbb{R}) \) be such \( \| \nu (\mathbb{R}) \| = \frac{1}{2} (\gamma_0) \) \# \nu (\mathbb{R}) \). Then \( \nu' + \mu' = \mu \) and \( \mu' = \mu' \). Thus \( \nu' + \mu' = \mu \) and \( \mu' = \mu' \). This case is probably false, in any case we do not have a proof.

**Conclusion:** \( \beta' \) is valid at \( a = b + 1 \) if and only if there is a strong sequence \( \beta' \) with \( \beta' = \beta' \) and \( \beta' = \beta' \).
Lemma: Suppose that \( \mathcal{F} \) is strong and \( \mathcal{F}(\pi) = \mathcal{F}(\pi') \) is an increased ultrapower with support \( S \). Then for each \( n \) either

1. \( \mathcal{F}(\pi) \) is maximal at \( n \), or
2. \( n = \pi(\pi') \) and \( \mathcal{F} \) is not maximal at \( \pi' \), or
3. for some \( s < \mathcal{F}(\pi) \), \( \mathcal{F}(\pi) \) factors onto \( \mathcal{F}(\pi') \) and \( \mathcal{F}(\pi) \) is the ultrapower of \( \mathcal{F}(\pi') \).

Proof: Suppose (1) fails and let \( S \) be a \( \mathcal{F}(\pi) \)-ultrafilter. Show the existence of \( D \) only depends on \( \mathcal{F}(\pi) \); \( \mathcal{F}(\pi) \) we can assume \( \mathcal{F} \) to fail and that \( \mathcal{F}(\pi) \) is bounded. By Lemma 7.4, there are maps

\[
\mathcal{F}(\pi) \rightarrow \mathcal{F}(\pi') \rightarrow \mathcal{F}(\pi)
\]

such that if \( \pi' = \pi(\pi') \) then \( \pi' \) is in \( \mathcal{F}(\pi) \).

Case 1 of (i) (Case (ii)). In this case we show that clause (iii) holds. If it does not then for some \( \pi \in \mathcal{F}(\pi) \) and \( \pi' \in \mathcal{F}(\pi') \), \( \pi \) is definable in \( \mathcal{F}(\pi) \); \( \pi \) is in \( \mathcal{F}(\pi) \) from members of \( \pi \) of \( \pi \) of \( \mathcal{F}(\pi) \), where \( \pi' \) is definable in \( \mathcal{F}(\pi) \) from members of \( \pi \) of \( \pi \) of \( \mathcal{F}(\pi) \). But \( \mathcal{F}(\pi') = \mathcal{F}(\pi') \) so \( \mathcal{F}(\pi) = \mathcal{F}(\pi') \), contradicting the fact that \( \mathcal{F}(\pi) \) is in \( \mathcal{F}(\pi') \).

Note that if clause (iii) holds then \( \mathcal{F}(\pi) \) satisfies the given conditions on \( \pi \). Hence by Lemma 7.5 it must actually be equal to \( \pi \).
Lemma 3.11.1. Let $U = \{ x \in \Sigma : (x) \in \emptyset \}$. We will show that $\emptyset$ is a $\mathcal{E}(G, K) \vdash (\emptyset)$-stable set, as claimed in the statement.

Suppose $f(x) = a'$ and $[f(x), x] \in \emptyset$. Then $f \vdash f(x)$ for some $f$ definable from members of $\Gamma \cup \Delta$. Then $\nu(f(x)) = \psi(f(x))$ and $\kappa(f(x)) = \xi(f(x))$, so $[\xi(f(x)) \vdash f(x)] \in \emptyset$. Hence $\xi(f(x)) = f(x)$, and so $\emptyset$ is normal.

If $[f(x), x] \in \emptyset$, then a similar argument shows that $[f(x), f(x)] = \emptyset$ for some $f \in \mathcal{E}(G, K)$ and $\emptyset(\mathcal{E}(G, K), \emptyset) = \mathcal{E}(G, K)$. We will show that there is $f \in \mathcal{E}(G, K)$ such that $f = [f(x)]$.

We want $\emptyset$ to be coherent and $\emptyset = \mathcal{E}(G, K)$, if $\emptyset$ is definable in $\mathcal{E}(G, K)$ from parameters in $\Gamma \cup \Delta$. Let $\psi$ be a formula and let $\mathcal{E}(G, K) \vdash \psi$ and $\emptyset \in \mathcal{E}(G, K)$ be parameters such that for all $\psi \in \mathcal{E}(G, K)$, $\mathcal{E}(G, K) \vdash \psi$ and $\mathcal{E}(G, K) \vdash \emptyset$. Then it is true in $\mathcal{E}(G, K)$ that there exists $\exists \in \mathcal{E}(G, K)$ such that if the function $f$ is defined by $f(x) = \exists(x)$, then $f(x) \in \emptyset \in \mathcal{E}(G, K)$ and $f(x) \in \emptyset$. Since $[f(x), x] = \emptyset$, it follows that in $\mathcal{E}(G, K)$ there is $\emptyset \in \mathcal{E}(G, K)$ such that if $f$ is defined by $f(x) = \exists(x)$ and $f(x) \in \emptyset$, then $f(x) \in \emptyset \in \mathcal{E}(G, K)$ and $f(x) \in \emptyset$. But then $[f(x), x] = \emptyset$, so $\emptyset$ is coherent.

Finally, $\mathcal{E}(G, K, \emptyset)$ is well-founded because it can be embedded in $\mathcal{E}(G, K) \vdash \emptyset$, which is well-founded. Since $\mathcal{D}$ is in full it follows from Theorem 3.11 that $\emptyset$ is absolutely well-founded.

In Section 1 we proved Theorem 3.11 (under the added hypothesis that $\mathcal{D}$ is countably complete for each pair $(\emptyset, \emptyset)$). In the next two results...
we eliminate this added hypothesis. A sequence $\mathfrak{M}$ is said to be normal if it is normal at all ordinals $\alpha$, that is, for all $\alpha$ there is a $\mathfrak{M}(\alpha)$-ultrafilter.

2.1 Lemma: There is a maximal sequence $\mathfrak{M}$: this sequence is strong and is unique.

Proof: The existence of $\mathfrak{M}$ is an easy recursive construction (e.g., if $\mathfrak{M}(\alpha)$ is normal then there is a $\mathfrak{M}(\alpha)$-ultrafilter). If any sequence and ultrafilter $\mathfrak{M}(\alpha)$ is not equal to $\mathfrak{M}$, then by Lemma 7.11 the sequence is unique provided it is strong. Lemma 7.6 also implies that the value of choice is not needed for this construction (e.g., provided the sequence is strong). In order to show that $\mathfrak{M}$ is strong we can construct a sequence $\mathfrak{M}_0$ which is strong by Lemma 13 and then show that there is an elementary embedding of $V_{\mathfrak{M}_0}$ into $V_{\mathfrak{M}}$ taking $\mathfrak{M}_0$ to $\mathfrak{M}$.

Claim: There is a full sequence $\mathfrak{M}_0$ such that for all $\alpha$, if $\mathfrak{M}(\alpha) > 0$ then $\mathfrak{M}(\alpha) = \mathfrak{M}_0$, and there is an $\mathfrak{M}_0$-closed, uncountable class $\mathcal{C}$ of cardinals $\alpha$ such that $\mathfrak{M}_0$ is normal at $\alpha$.

Proof: Start with a sequence $\mathfrak{M}_0$ which is maximal for countable successors of $\aleph_\alpha$; that is, such that there is no countably compact $\mathfrak{M}_0(\alpha+1)$-ultrafilter for any ordinal $\alpha$. Then for any $\alpha$ with cofinality equal to $\omega$ we have $\mathfrak{M}(\alpha) > 0$, and $\mathfrak{M}_0$ has cofinality greater than $\alpha$. It follows that any $\mathfrak{M}_0(\alpha+1)$-ultrafilter is countably compact, and hence $\mathfrak{M}_0$ is normal at such $\alpha$. Also, take an initial sequence $\mathfrak{M}(\alpha+1) = \mathfrak{M}(\alpha)$ as in Lemma 1.3 so that $\mathfrak{M}_0(\alpha+1)$ is normal for all $\alpha$ such that $\mathfrak{M}_0(\alpha) > 0$. Let $\mathfrak{M}_0$ be the class of cardinals $\alpha$ such that $\mathfrak{M}_0(\alpha) = \mathfrak{M}_0$, and $\mathfrak{M}_0(\alpha) > 0$ for all $\alpha < \alpha$. Then $\mathfrak{M}_0$ is $\mathfrak{M}_0$-closed uncountable: we will...
show that $q_i$ is maximal at $x_i$ if $x_i$ is in $Q_i$. If $x_i = y_i$ then $Q_i$ is maximal at $x_i$ by Lemma 7.8, since $Q_i$ is maximal at $y_i$. If $x_i \neq y_i$ then $x_i$ is not in the range of $i$, so by Lemma 7.8, $Q_i$ is maximal at $x_i$ unless $i$ includes an antecedent by an antecedent $d$ of $i$. But $d$ doesn't include such an antecedent, since $r(d) = x_i$, and $x_i$ was constructed by taking steps through a sequence of ordinals with cofinality different from $x_i$. (1) (Case

We will prove that $K^i_0$ is an elementary substructure of $K^i$ by defining a class $Z$ of elements such that $K^i_0 = K^i$ in isomorphic to the split word of $K$ in $K^i$). Let $\mathcal{X}$ be the first $n$ members of $Z$. The set $\mathcal{X}$ will be defined by induction on $z$ together with a class $\mathcal{Y}$ of ordinals such that $z \in \mathcal{Y}$, $\mathcal{X}$ will satisfy the following 3 conditions:

1. $(\forall y) \ (\forall x \in K^i_0 \cup \mathcal{X}) \ (y < x \Rightarrow y \in \mathcal{X})$, where $\mathcal{X} \cup \mathcal{Y}$ is the split word of $\mathcal{X}$.

2. $\mathcal{X}$ is thin.

3. The order type of $\mathcal{X}$ is $\omega$.

4. If $z = z'$, then $\mathcal{X} = \mathcal{X}'$.

5. If $z$ is a limit ordinal, then $\mathcal{X} \subseteq \mathcal{Y}$.

These conditions ensure that $K^i_0 = K^i_0$, so $\mathcal{X}$ is strong.

We set $\delta = \delta$ and $\delta = \delta$. If $\delta$ and $\delta$ have been defined, and $\delta$ is the least ordinal greater than $\delta'$ such that $\delta' \delta = 0$, then we define $\mathcal{X}$ and $\mathcal{Y}$ by letting $\mathcal{X}$ be the first $\delta$ members of $\mathcal{X}$ and $\mathcal{Y}$ be the first $\delta$ members of $\mathcal{Y}$. Conditions (1) - (4) are satisfied, and $\delta' = \delta' \delta' = \delta' \delta' \delta'$. If $\delta' \delta' = \delta'$, then $\delta' \delta' = \delta'$. (\delta' \delta' = \delta'
and $\mathcal{B}(x') = \emptyset$ as well by the maximality of $\mathcal{B}_x$. Hence $\mathcal{B}_x, \mathcal{B}_y, \mathcal{B}_z$ and conditions (3) is satisfied. If $\mathcal{F}$ is a limit of measurable ordinals then we can set $\mathcal{B}_x, \mathcal{B}_y, \mathcal{B}_z$ and $\mathcal{B}_w = \emptyset$. Now we are left with the only difficulty case, defining $\mathcal{B}_x$, and $\mathcal{B}_y$, when $\mathcal{B}_x$ and $\mathcal{B}_y$ have been defined and $\mathcal{B}_z, \mathcal{B}_w = \emptyset$. To deal with this case we will define an auxiliary decreasing sequence of elements $\mathcal{T}_n$. For each $\mathcal{T}_n$ combines (1)-(3) will hold with $\mathcal{B}_y$ replaced by $\mathcal{T}_n$. In addition, if $\mathcal{T}_n$ is the least member of $\mathcal{T}_n$ then

(6) If $\mathcal{T}_n < \mathcal{T}_m$, then $\mathcal{T}_n < \mathcal{T}_m$.

If $\mathcal{T}_n = \emptyset$ then we set $\mathcal{T}_n$ equal to $\mathcal{B}_y$, and if $\mathcal{T}_n$ is a limit ordinal then $\mathcal{T}_n, \mathcal{T}_n < \mathcal{T}_n$. We are left with the problem of defining $\mathcal{T}_n$, given $\mathcal{T}_n$. For each $\mathcal{T}_n$ let

$$\mathcal{T}_n \in \mathcal{B}(x, y, z) \cap \mathcal{B}(x, y, z)$$

be the transitive collapse. Then $\mathcal{T}_n \in \mathcal{B}(x, y, z) \cap \mathcal{B}(x, y, z)$. Also, by Lemma 2,

$$\mathcal{T}_n = \mathcal{T}_n \in \mathcal{B}(x, y, z) \cap \mathcal{B}(x, y, z)$$

for any $\lambda \in \mathcal{B}(x, y, z) \cap \mathcal{B}(x, y, z)$. We must have $\mathcal{T}_n \in \mathcal{B}(x, y, z) \cap \mathcal{B}(x, y, z)$.

The filter $\mathcal{F} = \mathcal{B}(x, y, z)$ is a $\mathcal{B}(x, y, z)$-ultralimit on $\mathcal{F}$, and

$$\mathcal{F} \in \mathcal{B}(x, y, z) \cap \mathcal{B}(x, y, z)$$

is well founded since $\mathcal{F}$ is a well-founded filter (Definition 3.10). By Lemma 7.1 we can construct proper iterated ultrapowers $\mathcal{U}$ and $\mathcal{U}$ so that the diagram.
commut. Since \( \Gamma_n \) was thick and \( \rho_k(\cdot) = 0 \) if \( \rho_k(\cdot) \neq n \),
\[ V = \{ \beta : \rho_k(\cdot) = \beta(\cdot) \} \]
is thick. Now set \( \Gamma'_m = \rho_k(\cdot) \), \( \Gamma_m \) is thick and \( V \) is not clearly \( \text{Ref}(\Gamma_m) \cup \text{Ref}(\Gamma_m) \cup V \). Classes \( (k) \) and \( (l) \) hold for \( \Gamma_m \), because they hold for \( \Gamma_n \). Classes \( (k) \) and \( (l) \) to determine: \( x \in V \) since \( \rho_k(x) = \rho_l(x) \), so \( b_k = a_{kl} = \rho_k(x) \), and \( b_k < b_{kl} \). Since \( \Gamma_m \) satisfies conditions \( (1) \) and \( (2) \), this completes the definition of the sequence of classes \( \Gamma_m \).

Since \( G \), the class of ordinals of cofinality \( \omega \) where \( G \) is non-trivial, is \( \omega \)-closed and unbounded there must be an ordinal \( \gamma \in G \) such that \( \omega_\gamma = \gamma \) for all \( \gamma < \gamma \). We will show that the construction of the \( \mathcal{C} \) sequence must stop with \( \Gamma_\gamma \).

CLAIM: Let \( \nu \) be as above and let \( \gamma \) be any thick class. Then \( \nu \not\in \mathcal{C} \cup \mathcal{R} \).

Proof: Set \( \mathcal{Q} = \nu \mathcal{G}_\nu \) be the transitive collapse of \( \nu \mathcal{U} \mathcal{G} \), as \( \alpha \cup \beta = \nu \mathcal{U} \mathcal{G} \). Suppose that, contrary to the claim, \( \nu \not\in \mathcal{C} \cup \mathcal{R} \).

Then \( \nu \) is the first ordinal moved by \( p \). Since \( \gamma \) is thick, \( p_\gamma \in \mathcal{G}_\nu \) so we can define \( \rho_\nu(x) = \rho_\nu(x) \). If \( \mathcal{G}_\nu(\cdot) \) is well-founded then Lemma 4 applies that either \( \rho_\nu(x) = \rho_\nu(y) \) or \( \rho_\nu(x) \) is not well-founded at \( y \). But by assumption there is no model of \( \text{Ref}(\cdot) \). If \( \rho_\nu(x) = \rho_\nu(y) \) and \( \mathcal{G}_\nu(x) \) is well-founded at \( y \) so \( \rho_\nu(x) \) must not be well-founded. We will complete the proof of the claim by showing that \( \rho_\nu(x) \) really is well-founded.
Suppose \((\mathcal{G})^n\) is not well-founded. Then there is a sequence 
\((f'(x; y; z; w; v; u; t; s))\) of functions \(f'(x; y; z; w; v; u; t; s)\) in \(\mathcal{G}\) and a sequence \((g_{n+1} = f'(x; y; z; w; v; u; t; s))\) of ordinals \(g_n = f'(x; y; z; w; v; u; t; s)\) such that if \(g_n = f'(x; y; z; w; v; u; t; s)\) for all \(x; y; z; w; v; u; t; s\). Since \(g'\) is well-ordered, by

Lemma 7.1, (iterated ultrapowers) and \(\aleph_0\) mapping \(\mathcal{G}'\) and \(\mathcal{G}\), note \(\mathcal{G}'\) for some sequence \(k\):

\[
\frac{\mathcal{G}'}{\mathcal{G}}
\]

Let \(a_k \in \mathcal{G}'\) be such that \(a_k \in \mathcal{G}'\), where the brackets represent the equivalence class in the ultrapower \(k\). Then for each \(a_k\) in \(k\):

\[
\pi(a_k) = \pi([i, j, k, l, m, n, o, p, q, r, s] \in \mathcal{G}')
\]

\[
= \pi([i, j, k, l, m, n, o, p, q, r, s] \in \mathcal{G})
\]

(Note that to simplify the notation the support of the ultrapower has been omitted.) But this says that if \(1^{\mathcal{G}'} = \mathcal{G}\) is the ultrapower of \(\mathcal{G}\), whose support is the image of the support of \(1\) that \(1^{\mathcal{G}'}(\pi([i, j, k, l, m, n, o, p, q, r, s] \in \mathcal{G}))\) is a decreasing sequence of ordinals in \(\mathcal{G}\). Set \(\mathcal{G}\) as iterated ultrapower of \(\mathcal{G}\), it will be fixed.

To complete the proof of Theorem 7.3, we still need one more claim.
Note: If \( \gamma_1 < \gamma_2 \) then \( G(x_1 \cup \{ \gamma_1 \}) \cap \gamma_2 \subseteq \gamma_2 \).

Proof: If this fails then there is \( \gamma_2 \subseteq \gamma_2 \) and a term \( s \) such that
\[
\delta(s) \neq \emptyset \iff \{ x_i \mid \delta(x_i) \subseteq \delta(s) \}
\]
and \( \delta(s) \neq \emptyset \) and \( x_i \neq x_j \) for \( i \neq j \), and \( x_i \neq x_j \). Then
\[
\delta(s) \neq \emptyset \iff \{ x_i \mid \delta(x_i) \subseteq \delta(s) \}
\]
for such \( s \) and \( y \), then \( \delta(x_i) \neq \emptyset \). Now \( \delta(x_i) \neq \emptyset \) and \( \delta(x_j) \neq \emptyset \) for \( x_i \neq x_j \).

If \( \gamma_1 \subseteq \gamma_2 \cup \{ \gamma_1 \} \neq \gamma_2 \) or \( \gamma_2 \neq \gamma_2 \), then \( \gamma_1 \neq \gamma_2 \) or \( \gamma_2 \neq \gamma_2 \), so this is impossible.

Now by the first claim with \( \gamma \subseteq \gamma_1 \) we must have \( \gamma \subseteq \gamma \) or \( \gamma \subseteq \gamma \), for some \( \gamma \subseteq \gamma \). By the second claim it follows that \( \gamma \subseteq \gamma \). But since \( \gamma \subseteq \gamma \), we must have \( \gamma \subseteq \gamma \). Now if the

correspondence does not hold \( \gamma \), then \( \gamma \gamma \) is defined and \( \gamma \subseteq \gamma \).

By the second claim \( \gamma \subseteq \gamma \gamma \gamma \), but this contradicts the first

case.

We could easily modify this argument to prove that \( \gamma \gamma \gamma \) is full, but

that is obvious from the next theorem:

**Theorem:** If \( D \) is any strong R sequence then there is an \( \mathcal{E} \) sequence

\( \mathcal{E} \Delta \mathcal{E} = \emptyset \).

If \( D \) is strong but not full then there is an \( \mathcal{E} \) sequence

\( \mathcal{E} \Delta \mathcal{E} = \emptyset \).

Proof of Theorem: If \( \mathcal{E} \Delta \mathcal{E} = \emptyset \) has been defined. If \( \mathcal{E} \Delta \mathcal{E} = \emptyset \).

Otherwise by Lemma 7.4 there must be \( \mathcal{E} \) such that

\( \mathcal{E} \Delta \mathcal{E} = \emptyset \).

Now if

\( \mathcal{E} \Delta \mathcal{E} = \emptyset \) then \( \mathcal{E} \Delta \mathcal{E} = \emptyset \).

Now if
Case (ii) of Lemma 7.9 holds, but this is impossible: it follows that
\[ h_i = \delta_i(h_i) \leq h_i \leq \delta_i \] for some \( i < \omega \) by the construction
\[ \delta_i(h_i) = h_i \leq h_i. \]
Hence \( h_i = \delta_i(h_i) \geq \delta_i(h_i) \) and \( \delta_i(h_i) \) is defined to be the supremum by \( \delta_i(h_i) \). It is easy to see that \( h_i \) is an upper bound such that \( h_i \geq h_i \). If \( i \) is proper and \( \delta_i(h_i) = h_i \) if \( i \) is improper. If \( i \) is full then \( i \) cannot be

\[ \delta_i \]

It is the first occurrence of the theorem is true. Since there does not exist a distinct sequence it follows that \( \delta_i \) is full. Hence if \( i \) is not null \( i \) cannot be proper, as the second occurrence of the theorem states.

**Remark:** Theorem 7.9 and Theorem hold if for some \( i \) on \( \delta_i \) mutual only at cardinality \( a \) such that \( \delta_i(a) = a \) and \( \delta_i(a) = a \) otherwise, provided that \( \delta_i(a) > a \). It follows that \( \delta_i(a) > a \) whenever \( \delta_i(a) > a \).

The proof of 7.9 is unfinished, so the proof of 7.0 never used to consider the case where case 7.9 (ii) holds: i.e., \( \delta_i \) is not mutual at \( a \), hence \( \delta_i \neq \delta_i(a) \) and \( \delta_i \) is not mutual at \( a \). In this case, \( \delta_i(a) \geq 2 \) or if \( \delta_i(a) = a \) in \( K \), then \( \delta_i(a) = a \) and \( \delta_i(a) = 0 \). Then \( \delta_i(a) = 0 \) is not mutual at \( a \), so \( \delta_i(a) = a \) if \( \delta_i(a) = a \) and \( \delta_i(a) = 0 \) as well, contradicting the choice of \( a \).

This completes our survey of the basic properties of \( \delta(a) \), we end with illustrations of the use of the theory in giving models with large cardinals.

**Full Paper:** Any of the following imply that there is no inner model of
\[ M \models \delta(a) = a^+ \].

(1) \( a \) and \( a^+ \) are both weakly compact.
Lemma 7.11. If there is no model of \( \mathcal{M}(p) \times \mathcal{M}(q) \), then there is no elementary embedding \( \mathcal{M}(p) \to \mathcal{M}(q) \) such that \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary. If \( \mathcal{M}(p) \to \mathcal{M}(q) \) is elementary, and \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary, then \( \mathcal{M}(p) \) is elementary.

Proof: Suppose \( \mathcal{M}(p) \times \mathcal{M}(q) \) is non-elementary. Then there is an elementary embedding \( \mathcal{M}(p) \to \mathcal{M}(q) \) such that \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary. If \( \mathcal{M}(p) \to \mathcal{M}(q) \) is elementary, then \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary. If \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary, then \( \mathcal{M}(p) \) is elementary.

Note that Lemma 7.11 applies in the case of the union of choice. We will prove Theorem 7.12 from more basic lemmas.

Lemma 7.13. If there is no model of \( \mathcal{M}(p) \times \mathcal{M}(q) \), then there is no elementary embedding \( \mathcal{M}(p) \to \mathcal{M}(q) \) such that \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary. If \( \mathcal{M}(p) \to \mathcal{M}(q) \) is elementary, and \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary, then \( \mathcal{M}(p) \) is elementary.

Proof: Suppose \( \mathcal{M}(p) \times \mathcal{M}(q) \) is non-elementary. Then there is an elementary embedding \( \mathcal{M}(p) \to \mathcal{M}(q) \) such that \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary. If \( \mathcal{M}(p) \to \mathcal{M}(q) \) is elementary, then \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary. If \( \mathcal{M}(p) \times \mathcal{M}(q) \) is elementary, then \( \mathcal{M}(p) \) is elementary.

Note that all of \( \mathcal{M}(p) \) is used to ensure that \( \mathcal{M}(p) \) is absolutely well-founded. If \( \mathcal{M}(p) \) is known to be absolutely well-founded, then \( \mathcal{M}(p) \times \mathcal{M}(q) \) is all that is needed. This fact is used in the next proof.

Proof of Lemma 7.14: Suppose \( \mathcal{M}(p) \) and \( \mathcal{M}(q) \) are both weakly compact. Since \( \mathcal{M}(p) \) is a special Amoeba, there is a consistent \( \mathcal{M}(p) \) such that \( \mathcal{M}(p) \times \mathcal{M}(q) \) is a subset of \( \mathcal{M}(p) \times \mathcal{M}(q) \). In particular, if \( \mathcal{M}(p) \times \mathcal{M}(q) \) is \( \mathcal{M}(p) \times \mathcal{M}(q) \), then \( \mathcal{M}(p) \times \mathcal{M}(q) \) is a subset of \( \mathcal{M}(p) \times \mathcal{M}(q) \). Again, \( \mathcal{M}(p) \times \mathcal{M}(q) \) is a subset of \( \mathcal{M}(p) \times \mathcal{M}(q) \), so by the weak compactness of \( \mathcal{M}(p) \) there is a \( \mathcal{M}(p) \) such that \( \mathcal{M}(p) \times \mathcal{M}(q) \) is a subset of \( \mathcal{M}(p) \times \mathcal{M}(q) \).
complete ultrapowers \( \mathcal{U} = P(\mathbb{R}) \cap L(\mathbb{R}) \). Then there is an elementary embedding \( i : L(\mathbb{R}) \rightarrow L(\mathbb{R})' \). Now if \( \beta < \zeta \) and \( \zeta \) is a successor ordinal, then \( \mathcal{U}_{\beta} = \mathcal{U}_{\zeta} \cap L(\mathbb{R})' \) and \( \mathcal{U}_{\beta}'(\mathbb{R})' = \mathcal{U}_{\zeta}'(\mathbb{R})' \). Note \( \beta \) is a successor ordinal as defined in \( i(\alpha) \). Now \( \mathcal{U}_{\beta}'(\mathbb{R})' \cap L(\mathbb{R}) \) and \( \mathcal{U}_{\zeta}'(\mathbb{R})' \cap L(\mathbb{R}) \) are elementary substructures of \( \mathcal{U}'(\mathbb{R})' \cap L(\mathbb{R}) \) and \( \mathcal{U}(\mathbb{R})' \cap L(\mathbb{R}) \), respectively. Let \( \alpha \) be countably complete, \( \alpha \neq \beta \), and the result follows from \( \alpha \). Hence, there is a model of \( \Delta(\mathcal{U}, \mathcal{U}') \cap L(\mathbb{R})' \) and \( \Delta(\mathcal{U}, \mathcal{U})' \cap L(\mathbb{R})' \) for all \( \alpha \neq \beta \) in \( \mathcal{U}'(\mathbb{R})' \cap L(\mathbb{R})' \) such that \( \beta \neq \zeta \) is a successor ordinal.

The other claims of Theorem 7.5 all follow from the following Lemma:

**Lemma:** Suppose there is no model of \( \Delta(\mathcal{U}, \mathcal{U})' \cap L(\mathbb{R})' \). Then for each ordinal \( \alpha \) there is \( \beta < \alpha \) such that \( \beta \neq \zeta \) is a successor ordinal.

**Proof:** Let \( \alpha \) be any such embedding. We can assume \( \beta = \alpha(\mathbb{R}) \neq \zeta \) from above. Let \( \mathcal{U}_\alpha \) be the sequence which is maximal from \( \mathcal{U} \) with \( \mathcal{U}_\alpha \cap \mathcal{U} \neq \emptyset \) and has \( \mathcal{U}_\alpha \cap \mathcal{U} = \mathcal{U}_\alpha \cap \mathcal{U} \). Now since \( \mathcal{U}_\alpha \cap \mathcal{U} \neq \emptyset \), \( \mathcal{U}_\alpha \) also has \( \mathcal{U} \cap \mathcal{U} \). Thus, by Theorem 7.5, there is a model of \( \Delta(\mathcal{U}, \mathcal{U})' \cap L(\mathbb{R})' \) and \( \mathcal{U}_\alpha \cap \mathcal{U} \cap \mathcal{U} \). Thus, there is an elementary embedding \( i : \mathcal{U}_\alpha \cap \mathcal{U} \rightarrow \mathcal{U}_\alpha \cap \mathcal{U} \cap \mathcal{U} \). Now it follows that \( \beta \) is a successor ordinal.

We have \( \Delta(\mathcal{U}, \mathcal{U})' \cap L(\mathbb{R})' \) is the transitive collapse of the universes in \( \mathcal{U}_\alpha \cap \mathcal{U} \). Then \( \beta \) is a successor ordinal as defined in \( \mathcal{U}_\alpha \cap \mathcal{U} \). Note \( \beta \neq \zeta \) is a successor ordinal as defined in \( \mathcal{U}_\alpha \cap \mathcal{U} \). In particular, \( \beta \neq \zeta \) as \( \beta \neq \zeta \) is a successor ordinal as defined in \( \mathcal{U}_\alpha \cap \mathcal{U} \).

We now have an elementary embedding \( i : \mathcal{U}_\alpha \cap \mathcal{U} \rightarrow \mathcal{U}_\alpha \cap \mathcal{U} \cap \mathcal{U} \) such that \( \beta \neq \zeta \) is a successor ordinal as defined in \( \mathcal{U}_\alpha \cap \mathcal{U} \). In addition, we have that \( \beta \) is a successor ordinal as defined in \( \mathcal{U}_\alpha \cap \mathcal{U} \).
If $f : B \rightarrow B$ is a map, then for all $x \in B$ if there exists a $y \in B$ such that $f(x) = y$, then $x = y$. 

We will first use the fact that $B \subseteq B$ to show that if $f(x) = x$, then $x = x$.

Suppose that $f(x) = x$ for all $x \in B$. Then by assumption $f(x) = x$.

We can find a sequence $(x_n)$ such that each $x_n$ is in $B$ and $x_n \to x$.

Let $x \in B$ be a limit point of $B$ and let $x_n \to x$ be a sequence of elements of $B$ cofinal in $x$. Then $x_n \to x$ for all $n \in \mathbb{N}$ and hence $x_n \to x$ is a sequence of elements of $B$ cofinal in $x$, which is impossible. The contradiction shows that $(x_n) \to x$. 

To find a bound $\beta \in B$ on $\|x\|_B$, we will have to refine this argument. Suppose $f$ is an inner automorphism of $B$, obtained by taking a map $\alpha$ on an interval $I$ to $I$. 

We can always render the ultrametric if necessary to ensure that $a \leq B$ if $a \leq b$. In order to make the argument cleaner, we will assume $f$ has this property. We will call $f$ weak if either $f$ is a limit point of $B$, or else $f$ is not a limit point of $B$. 

Let $x \in B$. Suppose $x \leq y$ for all $y \in B$. Then $x = y$. 

We now define a map $h$ inside $\mathbb{R}$, which is good in $\mathbb{R}$, by recursion on $x$. If we set $h(x) = h_B$ and if $h(x) = h_B$, then we have $h(x) = h_B$. 

We now define a map $h$ inside $\mathbb{R}$, which is good in $\mathbb{R}$, by recursion on $x$. If we set $h(x) = h_B$ and if $h(x) = h_B$, then we have $h(x) = h_B$. 

We now define a map $h$ inside $\mathbb{R}$, which is good in $\mathbb{R}$, by recursion on $x$. If we set $h(x) = h_B$ and if $h(x) = h_B$, then we have $h(x) = h_B$.
defined when \( h_{n,v} \cdot \delta_{R(v)} = \delta_{R(v)} \cdot h_{n,v} \) where \((n,v)\) is the
least pair with \( n < v \). For \( v' = v \) this will let \( v \) be good for \( k \) in
\( S_{R(v)} \). Since the construction is entirely inside \( S_{R(v)} \), the argument above
does that \( h(v) = 2^v \) in \( S_{R(v)} \). We set \( f = h(v) \), and it only remains to
draw that if \( v \) is any good traversed otherwise than \( j \) can be embedded into
\( \mathbb{N} \), so \( |j| \leq |\mathbb{N}| = c \). We still define an increasing map \( n(h) = h(v) \)
such that \( j \) is obtained by taking only the occurrences in the range of \( f \).
This will necessarily follow steps \( n \cdot \delta_{R(v)} = \delta_{R(v)} n \) so that
\( (\delta_{R(v)} n)(\delta_{R(v)} n)(\delta_{R(v)} n) \) = \( \omega(n,\omega) \) and the diagram below commutes:

\[
\begin{array}{c}
\delta_{R(v)}(n) & \xrightarrow{n(h)} & \delta_{R(v)}(n) \\
| & | & | \\
| & 1 & | \\
| & | & | \\
\delta_{R(v)} & \xrightarrow{n(h)} & \delta_{R(v)} \\
\end{array}
\]

Suppose \( \omega \) has been defined and \( \omega < \omega^{(\omega)} = \omega^{\omega} \) is less than
the length of \( k \). Then there is a map \( \delta_{R(v)}(\omega) = \delta_{R(v)}(\omega) \) and it is easy to
see that there is a \( \omega \) such that \( (\delta_{R(v)}(\omega))(\delta_{R(v)}(\omega))(\delta_{R(v)}(\omega)) \). We set \( \omega(\omega) = \omega \).

It is clear that \( \omega(\omega) = \omega \). Thus \( \omega(\omega) \) can be defined for all \( \omega \ \omega(\omega) \) so long as
\( \omega \) is less than \( |a(k)| \). Suppose \( \omega = a(k) \). If \( \omega(\omega) > \omega \) then
\( \omega(\omega) = \omega(\omega)(\omega(\omega)) \) is the next \( \omega \). Then \( \omega \) certainly has a suitable
greater than \( \omega \) in \( \delta_{R(v)}(\omega) \) so \( \omega = a(k) \). If \( \omega = \omega(\omega) \) then \( \omega(\omega) = \omega(\omega)(\omega(\omega)) \) is the next \( \omega \), and since \( j \) is good and \( \omega \) is a limit ordinal there is \( \omega < \omega \) such that
Theorem 7.20

(1) There is a sequence \( \langle a_n \rangle \) such that \( a_n = (a_n, b_n) \) and for all \( n \),

\[ a_n < a_m \text{ whenever } n < m. \]

Since \( \langle a_n \rangle \) is an unbounded sequence, \( a_n \) will converge to \( a \), the least \( a \) such that \( a_n < a \) for all \( n \). If \( a_n \) satisfies (1) then \( a_n \) is an element of the least \( f \) satisfying (1), where \( a_n = \sigma(a_n) \). But this statement can be proved by a weak induction on \( a_n \).

Proof of 7.18: We have already proved (1). The hypotheses of (1), (1a),

and (c) immediately imply that there is a sequence \( \langle c_n \rangle \) with \( c_n < c_{n+1} \) and \( c_n \to c^* \) in \( \mathcal{K}(\mathcal{X}) \) and hence imply the existence of a model of

\( \mathcal{I}(\mathcal{X}) = c^* \) by Lemma 7.15. Suppose \( c^* \) is a \( \mathcal{K} \)-complete ultrafilter on \( f \) and \( \mathcal{K}(f) = c^* \). Then \( \mathcal{K}(f) \) satisfies the axiom of choice,

\[ \mathcal{K}(f) \, \mathcal{K}(f) = c^* \] in \( \mathcal{K}(\mathcal{X}) \) as \( \mathcal{K}(f) \cap c^* \mathcal{K}(\mathcal{X}) = c^*. \]

Then Lemma 7.13 implies that \( \mathcal{I}(f) = c^* \). We follow the same procedure as in proving (11) by Lemma 3 of 7.11. If every \( \mathcal{K} \)-complete filter over \( \mathcal{X} \) can be extended to an ultrafilter then

for each \( k \in \mathcal{X} \) there is a ultrafilter \( c^* \) such that \( \mathcal{I}(f) = c^* \). Thus

(11) follows from Lemma 7.13.

Theorem 7.21

(11) \( \mathcal{I}(f) = c^* \). Let \( f \) be a \( \mathcal{K} \)-complete ultrafilter ideal over \( \mathcal{X} \). We can assume that \( f \) is normal (7.6).

Let \( P \) be the notion of forcing with conditions \( a \in \mathcal{X} \) such that \( a \notin f \) and with \( a \in f \) if \( f \) is normal. Then \( f \) is a \( \mathcal{K} \)-complete ultrafilter over \( \mathcal{X} \). Then \( f \) is the first ordinal numbered by \( \langle \gamma, \varphi \rangle \) for each \( \gamma \), and \( \mathcal{K} \) is well-founded. Since \( \mathcal{K} \) is a successor, \( \gamma \in \mathcal{X} \) for each \( \gamma \), we have \( \mathcal{K}(c) = c \mathcal{K}(\mathcal{X}) \). Since \( f \) is \( \mathcal{K} \)-complete, \( c^* \) is a ppoint in \( \mathcal{I}(c) \), so \( \mathcal{K}(c) \leq c^* \). But \( \mathcal{K}(c) \notin c^* \), so \( c^* \) is a \( \mathcal{K} \)-complete ultrafilter over \( \mathcal{X} \). Then \( \mathcal{I}(f) = c^* \). Thus every \( \mathcal{K} \)-complete ultrafilter ideal over \( \mathcal{X} \) is normal (7.6).
closed under countable sequences in \( \mathbb{N}(\mathbb{V}) \), as well as in \( \mathbb{V} \) and then the sequential condition \( \mathcal{E}_\lambda \) as defined in \( \mathbb{N}(\mathbb{V}) \), in the same as in \( \mathcal{V} \). Now if \( \mathcal{C}, (\alpha, \eta) \in \mathcal{V} \) is a sequence of numbers of \( \mathbb{N} \) then (see (50))) there is a sequence \( (\alpha, \eta) \in \mathcal{V} \) such that \( \mathcal{C}, \eta \mathcal{E}_\lambda [\lambda] \). Then \( \mathcal{C}, \eta \mathcal{E}_\lambda [\lambda] \Rightarrow \exists \eta \in \mathcal{V} \), so \( \mathcal{C} \subset \mathcal{V} \). The other question is interesting enough to tolerate us a separate lemma, which concludes the proof of Theorem 7.11.

**Lemma:** If \( \mathcal{C} \) is a set generic extension of \( \mathbb{V} \) than (1) for all \( \mathcal{C} \in \mathcal{V} \), \( \mathcal{C} \mathcal{E}(\mathcal{C}_{\mathcal{V}}) = \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \) and (2) \( \mathcal{C}_{\mathcal{V}} \) is still a maximal sequence in \( \mathcal{V}(\mathcal{V}) \).

**Proof:** Let \( \mathcal{C} \in \mathcal{V} \) be \( \mathcal{V} \)-generic over \( \mathcal{V} \) and suppose \( \mathcal{C} \mathcal{E}(\mathcal{V}) \). Then\( \mathcal{C} \mathcal{E}_{\mathcal{V}}(\mathcal{V}) \) as \( \mathcal{V} \)-generic over \( \mathcal{V} \). It follows that \( \mathcal{C}_{\mathcal{V}} \) is still a maximal sequence in \( \mathcal{V}(\mathcal{V}) \), since if \( \mathcal{C}_{\mathcal{V}} = \mathcal{V}(\mathcal{V}) \), \( \mathcal{V} \)-generic over \( \mathcal{V} \) in a single cardinal \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \), any other \( \mathcal{C}_{\mathcal{V}} \) would collapse \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \), so there are no \( \mathcal{C}_{\mathcal{V}} \) above. This is for any \( \mathcal{V} \) and hence no new \( \mathcal{V} \)-generic for \( \mathcal{C} \).

Now Lemma 7.4 has the hypothesis that there is no model of \( \mathcal{C} \mathcal{V}(\mathcal{V}) \), \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \), and it is not immediately obvious that this is true in \( \mathcal{V}(\mathcal{V}) \). However the proof of 7.5 only used the existence of a full sequence, and we do know that \( \mathcal{C}_{\mathcal{V}} \) is still full. Note if \( \mathcal{C}_{\mathcal{V}} \) is not maximal in the situation \( \mathcal{V}(\mathcal{V}) \) then at least the \( \mathcal{V}(\mathcal{V}) \)-ultrafilter \( \mathcal{U} \) is unique, but it is still unique in \( \mathcal{V}(\mathcal{V}) \). If \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \) is \( \mathcal{V} \)-generic over \( \mathcal{V} \), if \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \) then let \( \mathcal{V} = \mathcal{C}(\mathcal{V}) \). Then \( \mathcal{C} \) and \( \mathcal{V} \) are \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \)-ultrafilters in \( \mathcal{V}(\mathcal{V}) \), so \( \mathcal{C} = \mathcal{V} \). Then \( \mathcal{C} \) cannot be a \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \)-ultrafilter. Hence \( \mathcal{C} \mathcal{C}_{\mathcal{V}}(\mathcal{V}) \) in the ground model \( \mathcal{V} \), contradicting the fact that \( \mathcal{C}_{\mathcal{V}} \) is maximal there.

7.14, 7.11
We conclude with two further lemmas which, while it is not needed for the
main results here, has proved extremely useful in later work. The proof is
made considerably significantly simpler by using the following lemmas, due to
Gold and Jensen (§6 lemmas 8,9) which was referred to in part 2 of this paper:

\textbf{7.10 Lemma:} Suppose that } M \text{ is a well-founded model of set theory.

\textit{If } M \longrightarrow M \text{ is an}] iterated ultrafilter and } M \longrightarrow N \text{ is another } E_n
\text{ elementary embedding, then } } (\xi^+, i) \text{ exist for all ordinals } \xi \in M.

\textbf{7.11 Lemma:} If there is no model of } 3 \text{ and } M^* \text{ then every elementary
embedding is } E_n \longrightarrow M \text{ into a well-founded model } M \text{ is an iterated
ultrafilter by measures in } E_n.

**Proof:** We will prove the lemmas assuming that } i \text{ is not a limit that is, there
is } \alpha \text{ such that } N = (i(\alpha)) \text{ and } \alpha \notin M. \text{ The complete lemmas
follows from this, for if } i \text{ is arbitrary then the maps } i, \alpha
\text{ are all set based and hence iterated ultrafilters. An initial segment of the

iterations of $k_i$ will map an initial segment of $\mathfrak{p}$ onto $\langle \mathfrak{p}^i \rangle^i = \mathfrak{p}^i \mathfrak{p}$. As I saw through the original three initial segments of the iterations $k_i$ will fit together to yield $\mathfrak{p}$.

Since $i$ is not fixed it is easy to see that $N = \mathfrak{p}(\mathfrak{p})$, that $\mathfrak{p}(\mathfrak{p})$ is strong and full. Thus by Lemma 7.10 there is an initial ultrafilter $\mathfrak{q}$ of $\mathfrak{p}(\mathfrak{p})$ and we only need to show that $i = j$. Since $i$ is not fixed, $i = i(i) = i$ in thick, and Lemma 7.10 implies that $i \in \mathfrak{p}(\mathfrak{p})$ $\epsilon_{i+1} = i$ for $i > 0$. Let $i' = \mathfrak{p}(\mathfrak{p}) \ni i_{i+1} = i \mathfrak{p}(\mathfrak{p})$.

Then $N$ is full, and so there is an initial ultrafilter $\mathfrak{q}$ of $\mathfrak{p}(\mathfrak{p})$ $\ni$ $W$. Thus $j' \ni \mathfrak{q}$ $\ni \mathfrak{p}$, and so must be the identity by the maximality of $\mathfrak{p}$. Thus $i'$ is the identity, which implies that $i = j$.

$\square$


[72] A. Poincaré,