The sharp for the Chang model is small

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Abstract

Woodin has shown that if there is a Woodin limit of Woodin cardinals then there is, in an appropriate sense, a sharp for the Chang model. We produce, in a weaker sense, a sharp for the Chang model using only the existence of a cardinal κ having an extender of length $\kappa^{+\omega_1}$.

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1 Introduction

The Chang model, introduced in [Cha71], is the smallest model of ZF set theory which contains all countable sequences of ordinals. It may be constructed as $L({}^{\omega}\Omega)$, that is, by imitating the recursive definition of the L_α hierarchy: setting $\mathbb{C}_0 = \varnothing$ and $\mathbb{C}_{\alpha+1} = \mathrm{Def}^{\mathbb{C}_\alpha}(\mathbb{C}_\alpha)$, but modifying the definition for limit ordinals α by setting $\mathbb{C}_\alpha = [\omega]^\alpha \cup \bigcup_{\alpha'<\alpha} \mathbb{C}_\alpha$. Alternatively it may be constructed, as did Chang, by replacing the use of first order logic in the definition of L with the infinitary logic L_{ω_1,ω_1}. We write $\mathbb{C}$ for the Chang model.

Clearly the Chang model contains the set $\mathcal{R}$ of reals, and hence is an extension of $L(\mathcal{R})$. Kunen [Kun73] has shown that the axiom of choice fails in the Chang model whenever there are uncountably many measurable cardinals; in particular the theory of $\mathbb{C}$ may vary, even when the set of reals is held fixed. We show that in the presence of sufficiently large cardinal strength this is not true. This work is inspired by an earlier unpublished result of Woodin, which states that if there is a Woodin limit of Woodin cardinals, then there is a sharp for the Chang model. Our result is not comparable to Woodin's: ours has a much weaker hypothesis, but a much weaker conclusion. Perhaps the most striking aspect of the new result is its characterization of the size of the Chang model: although the Chang model, like $L(\mathcal{R})$, can have arbitrary large cardinal strength coded into the reals, the large cardinal strength in the Chang model, above that of $L(\mathcal{R})$, is at most $o(\kappa) = \kappa^{+\omega_1} + 1$ even in the presence of large cardinals in V,

The next three definitions describe our notion of a sharp for $\mathbb{C}$. Following this definition and a formal statement of our theorem, we will more specifically discuss the differences between our result and that of Woodin.

As with traditional sharps, the sharp for the Chang model asserts the existence of a closed, unbounded class I of indiscernibles. In order to state the conditions on I, as Definition 1.3, we need two preliminary definitions:

Definition 1.1. Say that a subset B of a closed class I is *suitable* if (a) it is countable and closed, (b) every member of B which is a limit point of I of countable cofinality is also a limit point of B, and (c) it is closed under predecessors in I.

We say that suitable sequences B and B' are *equivalent* if they have the same order type, and agree about which of their members are successor members of I.

Note that if B is suitable and β' is the successor of β in B, then either β' is the successor of β in I, or else β' is a limit member of I and $\mathrm{cf}(\beta') > \omega$. Indeed clauses (b) and (c) of the definition of a suitable sequence are equivalent to the assertion that every gap in B, as a subset of I, is capped by a member of B which is a limit point of I of uncountable cofinality.

Definition 1.2. Suppose that T is a collection of constants and functions with domain in $[\kappa]^n$ for some $n < \omega$. Write $\mathcal{L}_T$ for the language with symbols $\{=, \in\} \cup T$ (identifying T with a corresponding set of constant and function symbols). A *restricted formula* in the language $\mathcal{L}_T$ is a formula such that every variable occurring inside an argument of a member of T is free in φ.

Definition 1.3. We say that there is a *sharp for the Chang model* $\mathbb{C}$ if there is a closed unbounded class I of ordinals and a set T of Skolem functions having the following three properties:

1. Suppose that B and B' are equivalent suitable sets, and let $\varphi(B)$ be a restricted formula. Then
$$\mathbb{C} \models \varphi(B) \iff \varphi(B').$$
2. Every member of $\mathbb{C}$ is of the form $\tau(B)$ for some term $\tau \in T$ and some suitable sequence B.
3. If V' is any universe of ZF set theory such that $V' \supseteq V$ and $\mathcal{R}^{V'} = \mathcal{R}^V$ then, for all restricted formulas φ
$$\mathbb{C}^{V'} \models \varphi(B) \iff \mathbb{C}^V \models \varphi(B).$$
for any $B \subseteq I$ which is suitable in both V and V'.

Note, in clause 3, that $\mathbb{C}^{V'}$ may be larger than $\mathbb{C}^V$. A sequence B which is suitable in V may not be suitable in V', as a limit member of B may have uncountable cofinality in V but countable cofinality in V'. However the class I, as well as the theory, will be the same in the two models.

Recall that a traditional sharp, such as $0^\sharp$, may be viewed in either of two different ways: as a closed and unbounded class of indiscernibles which generates the full (class) model, or as a mouse with a final extender on its sequence which is an ultrafilter.

From the first viewpoint, perhaps the most striking difference between $0^\sharp$ and our sharp for $\mathbb{C}$ is the need for external terms in order to generate $\mathbb{C}$ from the indiscernibles. From the second viewpoint, regarding the sharp as a mouse, the sharp for the Chang model involves two modifications:

1. For the purposes of this paper, a *mouse* will always be a mouse over the reals, that is, an extender model of the form $J_\alpha(\mathcal{R})[\mathcal{E}]$.
2. The final extender of the mouse which represents the sharp of the Chang model will be a proper extender, not an ultrafilter.

It is still unknown how large the final extender must be. We show that its length is somewhere in the range from $\kappa^{+(\omega+1)}$ to $\kappa^{+\omega_1}$, inclusive:

Theorem 1.4 (Main Theorem). *1. Suppose that there is no mouse $M = J_\alpha(\mathcal{R})[\mathcal{E}]$ with a final extender $E = \mathcal{E}_\gamma$ such that* $\mathrm{cf}(\mathrm{length}(E)) > \omega$ *and* $\mathrm{length}(E)$ *is at least $\kappa^{+(\omega+1)}$ in $J_\alpha(\mathcal{R})[\mathcal{E}]$. Then $K(\mathcal{R})^{\mathbb{C}}$, the core model over the reals as defined in the Chang model, is an iterated ultrapower (without drops) of $K(\mathcal{R})^V$; in particular, there is no sharp for the Chang model.*

2. Suppose that there is a model $L(\mathcal{R})[\mathcal{E}]$ which contains all of the reals and has an extender E of length $(\kappa^{+\omega_1})^{L(\mathcal{R})[\mathcal{E}]}$, where κ is the critical point of E. Then there is a sharp for $\mathbb{C}$.

This problem was suggested by Woodin in a conversation at the Mittag-Lefler Institute in 2009, in which he observed that there was an immense gap between the hypothesis needed for his sharp, and easy results in the direction of the lower bound in clause 1 at, for example, a model with a single measure. At the time I conjectured that the same argument might show that any extender model would provide a similar lower bound, but James Cummings and Ralf Schindler, in the same conversation, suggested as more likely the bound exposed in Theorem 1.4(1).

I would also like to thank Moti Gitik, for suggesting his forcing for the proof of clause 2 and explaining its use. I have generalized his forcing to add new sequences of arbitrary countable length. I have also made substantial but, I believe, inessential changes to the presentation; I hope that he will recognize his forcing in my presentation. Many of the arguments in this paper, indeed almost all of those which do not directly involve either the generalization of the forcing or the application to the Chang model, are due to Gitik.

1.1 Comparison with Woodin's sharp

Our notion of a sharp for $\mathbb{C}$ differs from that of Woodin in several ways. We will discuss them in roughly increasing order of importance. The first two are, I believe, inessential: (i) The theory of our sharp can depend on the set of reals, while the theory of Woodin's sharp does not; however the invariance of his theory is due to the presence or absence of large cardinals, not to the definition of the sharp. An appropriate analogy is with the sharp for $L(\mathcal{R})$, which is equivalent to the existence of a mouse over the reals having a measurable cardinal. Woodin has shown that the theory of this mouse stabilizes in the presence of a class of Woodin cardinals, and the same proof applies to our sharp for the Chang model.

(ii) Woodin's sharp is defined in terms of the infinitary language L_{ω_1,ω_1}, whereas ours uses only first order logic; however these two languages are equivalent in this context: since $\mathbb{C}$ is closed under countable sequences and $\mathbb{C}_\alpha \prec \mathbb{C}$ whenever α is a member of the class I of indiscernibles, the existence of our sharp implies that any formula of L_{ω_1,ω_1} is equivalent to a formula of first order logic having a parameter which is a countable sequence of ordinals.

The status of the next two differences is unclear at this point, and requires further study. (iii) Woodin's sharp allows for any countable subsequence of I, while we allow only sequences which have all of their gaps capped by a limit point of I of uncountable cofinality. This allowance is somewhat relaxed in Theorem 3.6, and may be sensitive to an improved choice of the set of terms for our sharp; however I believe that this patterning of the indiscernible sequences reflects basic information about the Chang model.

(iv) The notion of restricted formulas is entirely absent from Woodin's results: he allows the terms from T to be used as full elements of the language. I believe that further work, with a better choice of terms, will eliminate the need for restricted formulas. The failure of this conjecture would expose a major weakness in our notion of a sharp.

Finally, (v) Woodin's construction is stronger in a way which makes it somewhat orthogonal to our construction: he has observed, in a personal communication, that his construction yields a sharp for a much larger model $\mathbb{C}^+$ which includes, in addition to all countable sequences of ordinals, the non-stationary ideal on $\mathcal{P}_{\omega_1}(\lambda)$ for each regular cardinal λ. Thus, although his work does imply as a corollary that there is a sharp for $\mathbb{C}$, the two constructions are complementary rather than in competition.

It should be emphasized that, as indicated by clauses (iii) and (iv) above, the definition given in this paper of the notion of a sharp for the Chang model and, especially, the specific choice made of the set of terms in the language and of the class I of indiscernibles should be regarded as preliminary. Their ultimate resolution will probably depend on closing the gap in the hypotheses of the two parts of Theorem 1.4 to determine the exact large cardinal strength of the sharp of the Chang model.

1.2 Some basic facts about $\mathbb{C}$

As pointed out earlier, the Axiom of Choice fails in $\mathbb{C}$ if there are uncountably many measurable cardinals; however, the fact that $\mathbb{C}$ is closed under countable sequences implies that the axiom of Dependent Choice holds, and this is enough to avoid most of the serious pathologies which can occur in a model without choice. For life without Dependent Choice, see for example [GK12], which gives a model with surjective maps from $\mathcal{P}(\aleph_\omega)$ onto an arbitrarily large cardinal λ without any need for large cardinals.

The same argument that shows that every member of L is ordinal definable implies that every member of $\mathbb{C}$ is definable in $\mathbb{C}$ using a countable sequence of ordinals as parameters.

In the proof of part 1 of Theorem 1.4 we make use of the core model $K(\mathcal{R})$, defined inside $\mathbb{C}$, and in the absence of the Axiom of Choice this requires some justification. In large part the Axiom of Choice can be avoided in the construction and theory of the core model, since the core model $K(\mathcal{R})$ itself can be well ordered by using countably complete forcing to map the reals onto ω — a process which does not change the Chang model. However one application of the Axiom of Choice falls outside of this situation: the use of Fodor's pressing down lemma, the proof of which requires choosing closed unbounded sets as witnesses that the sets where the function is constant are all non-stationary. This lemma is needed in the construction of $K(\mathcal{R})$ in order to prove that the comparison of pairs of mice by iterated ultrapowers always terminates. However, this is not a problem In the construction of $K(\mathcal{R})$ in $\mathbb{C}$, as we can apply Fodor's lemma in V, which we assume satisfies the Axiom of Choice, to verify that all comparisons terminate.

The proof of the covering lemma involves other uses of Fodor's lemma; however we do not use the covering lemma.

1.3 Notation

We use generally standard set theoretic notation. We use Ω to mean the class of all ordinals (sometimes regarded as an ordinal itself). In forcing, the notation $p \parallel \varphi$ means that the condition p decides φ, that is, either $p \Vdash \varphi$ for $p \Vdash \neg\varphi$. If s is a condition in a forcing order P, then we write $P\|_s$ for the forcing below s, that is, $\{\, p' \in P \mid p' \leqslant p \,\}$. We use $h[B]$ for the range of h on B, that is, $h[B] = \{\, h(b) \mid b \in B \,\}$. We use $[X]^\kappa$ for the set of subsets of X of size κ. If η is a well order then we use $\operatorname{otp}(\eta)$ for its order type.

If E is an extender, then we write $\operatorname{supp}(E)$ for the support, or set of generators, of E. Typically we take this to be the interval $[\kappa, \operatorname{length}(E))$ where κ is the critical point of E; however we frequently make use of the restriction of E to a non-transitive[1] set of generators: that is, if $S \subseteq \operatorname{supp}(E)$ then we write $E\circlearrowleft S$ for the restriction of E to S, so $\operatorname{Ult}(V, E\circlearrowleft S) \cong \{\, i^E(f)(a) \mid f \in V \wedge a \in [S]^{<\omega} \,\}$. We remark that $\operatorname{Ult}(V, E\circlearrowleft S) = \operatorname{Ult}(V, \bar{E})$, where $\bar{E}$ is the *transitive collapse of* E, that is, the extender obtained from E by using the transitive collapse $\sigma\colon [\kappa, \operatorname{length}(\bar{E})) \cong \operatorname{supp}(E) \cap \{\, i(f)(a) \mid f \in V \wedge a \in [S]^{<\omega} \,\}$ and setting its constituant ultrafilters by defining $(\bar{E})_\alpha = E_{\sigma^{-1}(\alpha)}$. This identification is frequently used in this paper in arguments which seem to naturally use the restricted extender $E\circlearrowleft S$, but $E\circlearrowleft S$ is not a member of the model in question. The fact that the collapse $\bar{E}$ is a member the model justifies these arguments. This use will not always be explicitly stated.

We make extensive use of the core model over the reals, $K(\mathcal{R})$. However we make no (direct) use of fine structure, largely because we make no attempt to use the weakest hypothesis which could be treated by our argument. The reader will need to be familiar with extender models, but only those weaker than strong cardinal, that is, without the complications of overlapping extenders and iteration trees. For our purposes, a mouse will be a model $M = J_\alpha(\mathcal{R})[\mathcal{E}]$, where $\mathcal{R}$ is the set $\mathcal{P}(\omega)$ of reals and $\mathcal{E}$ is a sequence of extenders, and it generally can be assumed to be a model of Zermelo set theory (and therefore equal to $L_\alpha(\mathcal{R})[\mathcal{E}]$).

If $M = J_\alpha(\mathcal{R})[\mathcal{E}]$ is a mouse then we write $M|\gamma$ for $J_\gamma(\mathcal{R})[\mathcal{E}\upharpoonright\gamma]$, that is, the restriction of M to ordinals below γ without including the active extender (if there is one) $\mathcal{E}_\gamma$ with index γ. We most commonly use this as $N|\Omega$ when the model N is the result of an iteration of length Ω and the length α of N is greater than Ω.

[1] In this context, we regard $\operatorname{supp}(E) = [\kappa, \lambda)$ as "transitive" despite its omission of ordinals less than κ. We could equivalently, but less conveniently, use $\operatorname{supp}(E) = \operatorname{length}(E)$.

2 The Lower bound

The proof of Theorem 1.4(1), giving a lower bound to the large cardinal strength of a sharp for the Chang model, is a straightforward application of a technique of Gitik (see the proof of Lemma 2.5 for $\delta = \omega$ in [GM96]).

Proof of Theorem 1.4(1). The proof uses iterated ultrapowers to compare $K(\mathcal{R})$ with $K(\mathcal{R})^{\mathbb{C}}$. Standard methods show that $K(\mathcal{R})^{\mathbb{C}}$ is not moved in this comparison, so there is an iterated ultrapower $\langle M_\nu \mid \nu \leqslant \theta \rangle$, for some $\theta \leqslant \Omega$ defined by $M_0 = K(\mathcal{R})$, $M_\alpha = \operatorname{dir}\lim\{ M_{\alpha'} \mid \alpha'' < \alpha' < \alpha \}$ for sufficiently large $\alpha'' < \alpha$ if α is a limit ordinal, and $M_{\alpha+1} = \operatorname{Ult}(M^*_\alpha, E_\alpha)$ where E_α is the least extender in M_α which is not in $K(\mathcal{R})^{\mathbb{C}}$ and M^* is equal to M_α unless E_α is not a full extender in M_α, in which case M^*_α is the largest initial segment of M_α in which E_α is a full extender.

We want to show that (i) this does not drop, that is, $M^*_\alpha = M_\alpha$ for all α, and (ii) $M_\theta = K(\mathcal{R})^{\mathbb{C}}$.

If either of these is false, then $\theta = \Omega$ and there is a closed unbounded class C of ordinals α such that $\operatorname{crit}(E_\alpha) = \alpha = i_\alpha(\alpha)$. Since $o(\kappa) < \Omega$ for all κ it follows that there is a stationary class $S \subseteq C$ of ordinals of cofinality ω such that $i_{\alpha',\alpha}(E_{\alpha'}) = E_\alpha$ for all $\alpha' < \alpha$ in S. Fix $\alpha \in S \cap \lim(S)$; we will show that M_α contradicts the hypothesis of Theorem 1.4(1).

To this end, let $\langle \alpha_n \mid n \in \omega \rangle$ be an increasing sequence of ordinals in S such that $\bigcup_{n\in\omega} \alpha_n = \alpha$. An argument of Gitik (see [GM96, Lemma 2.3]) shows that the threads belonging to generators of E_α are definable in $\mathbb{C}$ using $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ as the only parameter. That is, there is a formula φ such that for all $\gamma < (\alpha^{+\omega})^{M_\alpha}$, the formula $\varphi(\vec{\alpha}, \vec{\beta}, \alpha, \gamma)$ is true in $\mathbb{C}$ if and only if $i_{\alpha_n,\alpha}(\beta_n) = \gamma$ for all but finitely many $n \in \omega$. If $\eta < \alpha^{+(\omega+1)}$ then this can be extended to all $\gamma < \eta$ by using the thread $\langle i^{-1}_{\alpha_n,\alpha}(\eta) \mid n < \omega \rangle$ as an additional parameter. Since $E_\alpha \notin \mathbb{C}$ it follows that $\operatorname{length}(E_\alpha) \geqslant \kappa^{+(\omega+1)}$, which contradicts the hypothesis of Theorem 1.4(1). $\square$

3 The upper bound

The proof of Theorem 1.4(2) will take up the rest of this paper except for some questions in the final Section 5.

The hypothesis of Theorem 1.4(2) is stronger than necessary: our construction of the sharp for $\mathbb{C}$ uses only a sufficiently strong mouse over the reals, that is, a model $M = J_\gamma(\mathcal{R})[\mathcal{E}]$ where $\mathcal{E}$ is an iterable extender sequence.

At this point we describe a generic procedure for constructing a sharp from a suitable mouse. For this purpose we will assume that M is a mouse satisfying the following conditions: (i) $|M| = |\mathcal{R}|$, definably over M, indeed (ii) there is an onto function $h\colon \mathcal{R} \to M$ which is the union of an increasing ω_1 sequence of functions in M, and (iii) M has a last extender, $E \in M$, such that $\operatorname{length}(E) = (\kappa^{+\omega_1})^M$. We can easily find such a mouse from the hypothesis of Theorem 1.4(2) by choosing a model N of the form $L_\gamma(\mathcal{R})[\mathcal{E}]$ with the last two properties and letting M be the transitive collapse of Skolem hull of $\mathcal{R} \cup \omega_1$ in N. At the start of section 4.1, where we begin the actual proof, we will specify more precisely what assumptions we make about the mouse, but we have made no effort to determine the weakest mouse for which our techniques work.

We add one further condition on M: (iv) the least measurable cardinal of M should be larger than θ^M, the least cardinal λ of M such that $\lambda \neq f[\mathcal{R}]$ for any function $f \in M$. Any mouse satisfying the conditions (i–iii) can be made to satisfy condition (iv) by iterating the least measurable cardinal past θ. The iteration map will not be an elementary embedding, but it will preserve conditions (i–iii).

This condition would allow us to assume the Continuum Hypothesis by using $M[g]$ instead of M, where g is a V-generic map collapsing $\mathcal{R}$ onto ω_1. Doing so would not add any new countable sequences and hence would not affect the Chang model. This will be needed in sections 4.6 and 4.7 in order to construct M-generic sets; however we will not do so until then: but the reader certainly may, if desired, assume that this has been done.

The following simple observation is basic to the construction:

Proposition 3.1. *The mouse M is closed under countable subsequences.*

Proof. By the assumption (ii) on M, any countable subset $B \subseteq M$ is equal to $h[b]$ for a function $h \in M$ and a set $b \subset \mathcal{R}$. Since M contains all reals, and any countable set of reals can be coded by a single real, $b \in M$ and thus $B \in M$. $\square$

As in the case of $0^\sharp$, we obtain the sharp for the Chang model by iterating the final extender E out of the universe.

Definition 3.2. We write $i_\alpha \colon M_0 = M \to M_\alpha = \mathrm{Ult}_\alpha(M, E)$. In particular M_Ω is the result of iterating E out of the universe, so that $i_\Omega(\kappa) = \Omega$.

Let $\kappa = \mathrm{crit}(E)$. We write $\kappa_\nu = i_\nu(\kappa)$ and $I = \{\, \kappa_\nu \mid \nu \in \Omega \,\}$. The *generators* belonging to κ_ν are the ordinals $i_{0,\nu}(\beta)$ such that $\kappa \leqslant \beta < (\kappa^{+\omega_1})^M$.

Note that every member of M_Ω is equal to $i_{0,\Omega}(f)(\vec{\beta})$ for some function $f \in M$ with domain κ and some finite sequence $\vec{\beta}$ of generators. The following observation follows from this fact together with Proposition 3.1.

Proposition 3.3. *Suppose that $N \supseteq M_\Omega | \Omega$ is a model of set theory which contains all countable sets of generators. Then $\mathbb{C}^N = \mathbb{C}$.*

Proof. It is sufficient to show that N contains all countable sets of ordinals, but that is immediate since every countable set B of ordinals has the form $B = \{\, i_\Omega(f_n)(\vec{\beta}_n) \mid n \in \omega \,\}$, where each f_n is a function in M and each $\vec{\beta}_n$ is a finite sequence of generators. The sequence $\langle\, f_n \mid n \in \omega \,\rangle$ is in $M \subseteq N$ by Proposition 3.1 and the sequence $\langle\, \vec{\beta}_n \mid n \in \omega \,\rangle$ is in N by assumption, so $B \in N$. $\square$

Clearly the class I gives a sharp for the model $M_\Omega | \Omega$ in the sense of Definition 1.3 (with suitable sequences from I replaced by finite sequences), but it is not at all clear that I gives a sharp for $\mathbb{C}$ as well. We show starting in section 3.3 that it does give a sharp when defined using the mouse specified there.

We conjecture that the sharp for the chang model can be obtained exactly in the same way as the sharp for other models:

Conjecture 3.4. Let M be the least $\mathcal{R}$-mouse such that $M \notin \mathbb{C}$. Then $K(\mathcal{R})^{\mathbb{C}}$ is the lower part (below Ω) of an iterated ultrapower of M, and M provides a sharp in (at least) the sense of this paper.

We will refer to this hypothetical mouse M as the "optimal" mouse. A verification of this conjecture would determine the correct large cardinal strength of the sharp, and could be expected to remove some of the weaknesses which have been remarked on in our results.

3.1 Why is suitability required?

The next result apparently shows that the restriction to suitable sets B in the Definition 1.3 of a sharp cannot be removed. The qualification "apparently" is needed because Proposition 3.5 does not apply to the sharp which we construct, but only to the optimal sharp which we conjecture exists. Nevertheless, Proposition 3.5 is an important motivation for our argument for Theorem 1.4(2).

Following Proposition 3.5, Theorem 3.6 is a strengthening of our Main Theorem 1.4(2) which comes close to suggesting that Proposition 3.5 is the only restriction (at least in this direction) to the sharp for $\mathbb{C}$.

The core model $K(\mathcal{R})^{\mathbb{C}}$ of the Chang model should not be expected to equal $M_{\Omega}|\Omega$, where M is the optimal mouse; rather it will be an iterate of that model. This is because all of the members of I are measurable in $M_{\Omega}|\Omega$, but it is likely that every measurable cardinal of $K(\mathcal{R})^{\mathbb{C}}$ has cofinality ω.

Proposition 3.5. *Assume that $K(\mathcal{R})^{\mathbb{C}}$ is an iterate of $M_{\Omega}|\Omega$, with an iteration map $k\colon M_{\Omega}|\Omega \to K(\mathcal{R})^{\mathbb{C}}$ such that $k(\kappa_\nu) = \kappa_\nu$ for all $\kappa_\nu \in I$. Suppose further that k is consistent with the set T of terms.*

Then for any two closed, countable subsets B and B' of I which disagree infinitely often about either (i) *which adjacent members are not adjacent in I or* (ii) *which members are limit points of I of uncountable cofinality, there is a restricted formula φ such that $\mathbb{C} \models \neg\big(\varphi(B') \iff \varphi(B)\big)$.*

We will not give a precise definition of the assumption that "k is consistent with the set T of terms"; however we will point out where it is used in the proof.

Proof. First suppose that $B = \langle \lambda_\nu \mid \nu < \xi \rangle$ and $B' = \langle \lambda'_\nu \mid \nu < \xi \rangle$ are counterexamples to clause (i). Thus they are increasing subsequences of I of the same length, and there is an infinite increasing sequence $\langle \nu_n \mid \nu < \omega \rangle$ of ordinals smaller than ξ such that for each n, λ_{ν_n+1} is the successor in I of λ_{ν_n}, but λ'_{ν_n+1} is larger than the successor in I of λ'_{ν_n}.

Let U_n be the ultrafilter on λ'_{ν_n+1} associated to the image of E on λ'_{ν_n+1}. That is, if ξ_n is the ordinal such that $\lambda'_{\nu_n+1} = \kappa_{\xi_n}$, then

$$U_n = i_{\xi_n}\big(\{\, x \subseteq \kappa \mid \kappa \in i^E(x) \,\}\big).$$

Now let τ_n be the least member of I above λ'_{ν_n}, so that $\lambda'_{\nu_n} < \tau_n < \lambda'_{\nu_n+1}$. Then $\langle \tau_n \mid n \in \omega \rangle$ is $M_\Omega|\Omega$-generic for the Prikry forcing with conditions

$$\{ (\vec{a}, \vec{A}) \mid \exists k < \omega(\forall i < k\, a_i < \lambda'_{i+1} \wedge \forall i \geqslant k\, A_i \in U_i) \},$$

which adds a single indiscernible for each of the ultrafilters U_n. In particular, there exists such a Prikry sequence $\langle c_n \mid n \in \omega \rangle$ such that $\lambda'_{\nu_n} < c_n < \lambda'_{\nu_n+1}$ for each $n \in \omega$. This fact is preserved by the iteration k: there is a $K(\mathcal{R})^{\mathbb{C}}$-generic Prikry sequence for the sequence of ultrafilters $k(U_n)$, each member of which lies in the interval $(k(\lambda'_{\nu_n}), \lambda'_{\nu_n+1})$. There is a restricted formula $\varphi(B')$ which asserts that there is such a sequence, provided that the terms of the language are consistent with k in the sense that there are terms τ_1 and τ_2 in T such that $\tau_1(\lambda'_{\nu_n}) = k(\lambda'_{\nu_n})$ and $\tau_2(\lambda_{\nu_n+1}) = k(U_n)$ for each $n < \omega$. Then $\varphi(B')$ is true in $\mathbb{C}$.

To see that this $\varphi(B)$ is false in $\mathbb{C}$, let c_n be any sequence with $k(\lambda_{\nu_n}) < c_n < \lambda_{\nu_n+1}$ for each n. Since λ_{ν_n+1} is the next member of I, the assumption on c_n implies that there is a function $f \in M$ such that $c_n < k \circ i_\Omega(f)(k(\lambda_{\nu_n})) < \lambda_{\nu_n+1}$ for each $n < \omega$. Since by assumption $k\colon M_\Omega|\Omega \to K(\mathcal{R})^{\mathbb{C}}$, the function $k \circ i_\Omega(f) \restriction \lambda_{\nu_n+1}$ is in $K(\mathcal{R})^{\mathbb{C}}$, and hence the sequence $\langle c_n \mid n < \omega \rangle$ is not generic over $K(\mathcal{R})^{\mathbb{C}}$. Thus $\varphi(B)$ is false in $\mathbb{C}$, and this completes the proof for clause (i).

Now suppose B and B' do not satisfy clause (ii): there is an infinite sequence $\langle \nu_n \mid n < \omega \rangle$ of ordinals below ξ such that for each $n < \omega$, λ_{ν_n} is a successor member of I but λ'_{ν_n} is a limit member of I of uncountable cofinality. Then the analysis given in the first part of the proof shows that if U_n is the ultrafilter on λ'_n, then for any sequence $(\upsilon_n \mid n < \omega)$ such that $\upsilon_n < \lambda'_{\nu_n}$ for all $n \in \omega$, there is a $K(\mathcal{R})^{\mathbb{C}}$-generic Prikry type sequence for the sequence of ultrafilters $\langle k(U_n) \mid n < \omega \rangle$ with the nth member in the interval $(\upsilon_n, \lambda'_{\nu_n})$. Again (assuming k is consistent with the terms of T) this statement can be made by a restricted formula, and that formula is false for B: Let τ_n be the immediate predecessor in I of τ_n. Then the argument given in the first part of this proof shows that there is no Prikry sequence having each member c_n in the interval $(k(\tau_n), \lambda_{\nu_n})$. $\square$

Note that while Proposition 3.5 says that gaps in B are significant, it does not attach any significance to the length of the gaps other than the distinction between gaps headed by a limit or successor member of member of I. Furthermore, it does not attach significance to individual gaps, but only to infinite sequences of gaps. The following strengthening of Theorem 1.4(2) can be proved by the technique used in Subsection 4.8 to deal with the special case $\kappa \notin B$ of Theorem 1.4(2).

Theorem 3.6. *Call B* weakly suitable *if B is a countable closed subsequence of I such that $B \cap \lambda$ is unbounded in λ whenever $\lambda \in B$ and* $\mathrm{cf}(\lambda) = \omega$. *Call two weakly suitable sequences B and B'* equivalent *if they have the same length and, with at most finitely many exceptions, corresponding successor members $\lambda \in B$ and $\lambda' \in B'$ satisfy* (i) *λ is a successor point of I if and only if λ' is, and* (ii) *if λ and λ' are successor members of I, then the I-predecessor of λ is in B if and only if the I-predecessor of λ' is in B'.*

Then Theorem 1.4(2) holds for weakly suitable sequences under this notion of equivalence.

3.2 Definition of the set T of terms.

The next definition gives the set of terms we will use to construct the sharp. This list should be regarded as preliminary, as a better understanding of the Chang model will undoubtedly suggest a more felicitous choice.

Definition 3.7. The set T of terms of the language for $\mathbb{C}$ are those terms obtained by compositions of the following set of basic terms:

1. For each function $f \in M$ (including a constant function) with domain and range contained in ${}^{<n}\kappa$, there is a term τ such that $\tau(z) = i_{0,\Omega}(f)(z)$ for all z for which the right side is defined.

2. For each β in the interval $\kappa \leqslant \beta < (\kappa^{+\omega_1})^M$ there is a term τ such that $\tau(\kappa_\nu) = i_{0,\nu}(\beta)$ for all $\nu \in \Omega$.

3. Suppose $\langle \tau_n \mid n \in \omega \rangle$ is an ω-sequence of compositions of terms from the previous two cases, and $\text{domain}(\tau_n) \subseteq {}^{k_n}\Omega_n$. Then there is a term τ such that $\tau(\vec{a}) = \langle \tau_n(\vec{a} \restriction k_n) \mid n \in \omega \rangle$ for all $\vec{a} \in {}^{\omega}\Omega$.

4. For each formula φ, there is a term τ such that if ι is an ordinal and y is a countable sequence of terms for members of $\mathbb{C}_\iota$ then
$$\tau(\iota, y) = \{\, x \in \mathbb{C}_\iota \mid \mathbb{C}_\iota \models \varphi(x, y) \,\}.$$

Proposition 3.8. *For each $z \in \mathbb{C}$ there is a term $\tau \in M$ and a suitable sequence B such that $\tau(B) = z$.*

Proof. First we observe that any ordinal ν can be written in the form $\nu = i_\Omega(f)(\vec{\beta})$ for some $f \in M$ and finite sequence $\vec{\beta}$ of generators. Each generator β belonging to some $\kappa_\xi \in i$ is equal to $i_\xi(\bar{\beta})$ for some $\bar{\beta} \in \big[\kappa, (\kappa^{+\omega_1})^M\big)$, and thus is denoted by a term $\tau(\kappa_\xi)$ built from clause (2). Thus any finite sequence of ordinals is denoted by an expression using terms of type (1) and (2). Since M is closed under countable sequences, adding terms of type 3 adds in all countable sequences of ordinals.

Finally, any set $x \in \mathbb{C}$ has the form $\{\, x \in \mathbb{C}_\iota \mid \mathbb{C}_\iota \models \varphi(x, y) \,\}$ for some ι, φ and y as in clause (4). Thus a simple recursion on ι shows that every member of $\mathbb{C}$ is denoted by a term from clause (4). $\square$

The terms specified in clause (2) force the limitation to restricted formulas in Theorem 1.4(2), since the domain of these terms is exactly the class I of indiscernibles. It is possible that a more natural set of terms would enable this restriction to be removed, but this would depend on a precise understanding of the iteration k.

By Proposition 3.8, every ordinal is denoted by a term from T using as parameters only members of $\alpha + 1$. This is contrary to the spirit of $0^\sharp$, where the term denoting α may require parameters from $I \backslash (\alpha+1)$. This seems to be a weakness in our current approach, and may suggest a direction for its refinement.

3.3 Outline of the proof

Proposition 3.3 suggests a strategy for the proof of Theorem 1.4(2): find a generic extension of $M_\Omega|\Omega$ which contains all countable sequences of generators. There are good reasons why this is likely to be impossible, beginning with the problem of actually constructing a generic set for a class sized model[2].

Beyond that, many of the known forcing constructions used to add countable sequences of ordinals require large cardinal strength far stronger than that assumed in the hypothesis of Theorem 1.4, and give models with properties which are known to imply the existence of submodels having strong large cardinal strength. However, two considerations suggest that this last problem may be less serious than it may appear. First, the Chang model may reflect more large cardinal strength than is apparent, since much of the large cardinal strength in V is encoded in the set of reals; and, second, many of the properties requiring the existence of models with large cardinals are false in the Chang model because of the failure of the Axiom of Choice. Results involving the size of the power set of singular cardinals, for example, are irrelevant to the Chang model since the power set is not (typically) well ordered there.

We avoid the problem of finding generic extensions of a class sized model by working with submodels generated by countable subsets of I, and we find that in fact none of the large cardinal structure in V survives the passage to the Chang model beyond that given in the hypothesis to Theorem 1.4.

Definition 3.9. If $B \subseteq I$, then we write

$$M_B = \{i_\Omega(f)(b) \mid b \text{ is a finite set of generators for members of } B\}.$$

If B is closed, and in particular if it is suitable, then we write $\mathbb{C}_B$ for the Chang model evaluated using the ordinals of $M_B|\Omega$ and all countable sequences of these ordinals.

Note that M_B is not transitive; it is a submodel of M_Ω, and $i_\Omega : M \to M_B$ is the canonical embedding for any $B \subseteq I$. The definition of $\mathbb{C}_B$ implies that if B and B' are closed sets with the same order type then $\mathbb{C}_B \cong \mathbb{C}_{B'}$. In particular, if $\operatorname{otp}(B) = \alpha+1$ then $\mathbb{C}_B \cong \mathbb{C}_{B(\alpha+1)} = \mathbb{C}_{\kappa_{\alpha+1}}$ where $B(\alpha+1) = \{\, \kappa_\nu \mid \nu < \alpha+1 \,\}$.

The motivation for our work began with the observation that $M_B|\Omega \prec M_{B'}|\Omega \prec M_\Omega|\Omega$ whenever $B \subseteq B' \subseteq I$. Proposition 3.5 refutes any suggestion that this necessarily extends to the models $\mathbb{C}_B$ and $\mathbb{C}_{B'}$, however it also motivates Definition 3.10 below.

That proposition says that we must take account of the gaps in B. To be precise, we will say that a *gap* in B is a maximal nonempty interval in $I \backslash B$. For

[2] There is the intriguing possibility that this could be done by *using* the existence of $\mathbb{C}^\sharp$.

all sets $B \subseteq I$ which we consider, every gap in B is headed by a limit point λ of I which is a member of $B \cup \{\Omega\}$ with uncountable cofinality.

Definition 3.10. A subset B of I is *limit suitable* if (i) its closure $\bar{B}$ is suitable, and every gap in B is an interval of the form $[\lambda, \delta)$ where (ii) δ is either Ω or a member of B which is a limit point of I of uncountable cofinality, (iii) $\lambda = \sup(\{0\} \cup B \cap \delta)$, and (iv) $\lambda = \kappa_{\nu+\omega}$ for some $\nu \in \Omega$.

Two limit suitable sets B and B' are said to be *equivalent* if they have the same order type and they have gaps in the same locations. If B is a limit suitable sequence then we write $\mathbb{C}_B$ for the Chang model constructed using only those countable sequences which are in a suitable subset $\tilde{B} \subset B$:

$$\mathbb{C}_B = L_{\Omega \cap M_B}(\mathcal{W}) \text{ where } \mathcal{W} = \bigcup \{\, [\Omega \cap M_{\tilde{B}}]^\omega \mid \tilde{B} \subseteq B \;\;\&\;\; \tilde{B} \text{ is suitable}]\,\}.$$

The use of $\kappa_{\nu+\omega}$ in clause (iv) is for convenience; our arguments would still be valid if it were only required that λ is a limit member of I of countable cofinality.

Note that if B is a limit suitable sequence then $\mathbb{C}_B$ is not closed under countable sequences; in particular B is not a member of $\mathbb{C}_B$. Thus if δ is the head of a gap of B then $\mathbb{C}_B$ believes (correctly) that δ has uncountable cofinality.

Theorem 1.4(2) will follow from the following lemma:

Lemma 3.11 (Main Lemma)**.** *If $B \subset I$ is limit suitable then $\mathbb{C}_B \prec \mathbb{C}$.*

Note that it is not obvious even that $\mathbb{C}_B \subseteq \mathbb{C}$, or, more accurately, that $\mathbb{C}_B$ is isomorphic to a subset of $\mathbb{C}$. The proof of Lemma 3.11 will use an induction on pairs (ι, φ), with $\iota \leqslant \Omega$, in which the induction hypothesis implies that the map $\sigma_\iota \colon \mathbb{C}_\iota^{\mathbb{C}_B} \to \mathbb{C}_\iota f$ defined by setting

$$\sigma_\iota \left(\{\, x \in \mathbb{C}_{\iota'}^{\mathbb{C}_B} \mid \mathbb{C}_{\iota'}^{\mathbb{C}_B} \models \varphi(x, a) \,\} \right) = \{\, x \in \mathbb{C}_{\iota'} \mid \mathbb{C}_{\iota'} \models \varphi(x, \sigma_\iota(a)) \,\}),$$

for each $\iota' < \iota$, $a \in \mathbb{C}_{\iota'}$, and formula φ of set theory, is an isomorphism between $\mathbb{C}_\iota^{\mathbb{C}_B}$ and a subset of $\mathbb{C}_\iota$.

To see that Lemma 3.11 suffices to prove Theorem 1.4(2), observe that any suitable set B can be extended to a limit suitable set defined by the equation

$$B' = B \cup \{\, \kappa_{\nu+n} \mid \kappa_\nu \in B \wedge n \in \omega \,\},$$

that is, by by adding the next ω-sequence from I at the foot of each gap of B and to the top of B. Now let B_0 and B_1 be two equivalent suitable sets. Then their limit suitable extensions B'_0 and B'_1 are also equivalent, having the same ordertype and having gaps in the corresponding places, so $\mathbb{C}_{B'_0} \cong \mathbb{C}_{B'_1}$. Then for any restricted formula φ we have

$$\begin{aligned} \mathbb{C} \models \varphi(B_0) &\iff \mathbb{C}_{B'_0} \models \varphi(B_0) \\ &\iff \mathbb{C}_{B'_1} \models \varphi(B_1) \iff \mathbb{C} \models \varphi(B_1). \end{aligned}$$

4 The Proof of the Main Lemma

The main tool used in this section is a forcing $P(\vec{E}\restriction\delta)/\leftrightarrow$, defined in M, such that $\mathbb{C}_B$ is definable in $M_B[G]$ where G is an M_B-generic subset of $i_\Omega(P(\vec{E}\restriction\delta)/\leftrightarrow)$ which can be constructed in $V[h]$ for any generic Levy collapse map $h\colon \omega_1 \cong \mathcal{R}$. The definition of the forcing and exposition of its properties will take up several sections. The actual proof of the lemma will be given (except for a special case treated in Section 4.8) in Section 4.7.

The forcing we use is essentially due to Gitik (see, for example, [Git02]) and the technique for constructing the M_B-generic set G is from Carmi Merimovich [Mer07]. Gitik's forcing was designed to make the Singular Cardinal Hypothesis fail at a cardinal of cofinality ω by adding many Prikry sequences, each of which is (in our context) a sequence of generators for a fixed ω-sequence of members of I. Thus it does what we need for the case when $\operatorname{otp}(B) = \omega$, but needs to be adapted to work for sequences B of arbitrary countable length. To this end we modify Gitik's forcing by using ideas based on Magidor's adaptation [Mag78] of Prikry forcing to add sequences of indiscernibles of cofinality greater than ω. This adds some complications to Gitik's forcing, but on the other hand much of the complication of Gitik's work is avoided since we do not have to avoid collapsing cardinals in the interval $(\kappa^+, \kappa^{+\omega_1})$, and hence can omit his preliminary forcing.

Our forcing is based on a sequence $\vec{E}$ of extenders, derived from the last extender E of M. We begin by defining this sequence, and at the same time specify what properties we require of the mouse M.

Definition 4.1. We define an increasing sequence, $\langle N_\nu \mid \nu < \omega_1 \rangle$ of submodels of M. We write E_ν for $E \cap N_\nu$, the restriction of E to the ordinals in N_ν, we write $\pi_\nu\colon \bar{N}_\nu \to N_\nu$ for the Mostowski collapse of N_ν, and we write $\bar{E}_\nu$ for $\pi_\nu^{-1}[E_\nu] = \pi_\nu^{-1}(E) \cap \bar{N}_\nu$.

We require that the $\mathcal{R}$-mouse M and the sequence $\langle N_\nu \mid \nu < \omega_1 \rangle$ satisfy the following conditions:

1. M is a model of Zermelo set theory such that $\mathcal{R} \subset M$, $|M| = |\mathcal{R}|$, and $\operatorname{cf}(\Omega^M) = \omega_1$.
2. $\operatorname{length}(E) = (\kappa^{+\omega_1})^M$.
3. If $\nu' < \nu < \omega_1$ then $(N_{\nu'}, E_{\nu'}) \prec (N_\nu, E_\nu) \prec (M, E)$.
4. ${}^\kappa N_\nu \cap M \subseteq N_\nu$.
5. $|\bar{N}_\nu|^M \subset N_\nu$.
6. [3] (i) $\kappa^{+(\omega+1)} \subseteq N_0$ and $M \models |\bar{N}_0| = \kappa^{+(\omega+1)}$, and (ii) for each $\nu > 0$ $(\sup_{\nu'<\nu} |\bar{N}_{\nu'}|)^{+(\omega+1)} \subseteq N_\nu$ and $M \models |\bar{N}_\nu| = (\sup_{\nu'<\nu} |\bar{N}_{\nu'}|)^{+(\omega+1)}$.
7. $M = \bigcup_{\nu<\omega_1} N_\nu$.

[3] This clause seems to be required for Definition 4.32.

We will work primarily with the extenders E_ν rather than with their collapses $\bar{E}_\nu$, because this makes it easier to keep track of the generators. However it should be noted that E_ν may not be a member of $\mathrm{Ult}(M,E)$, so that further justification is needed for many of the claims we wish to make about being able to carry out constructions inside M. Since we never actually use more than countably many of the extenders E_ν at any one time, the following observation will provide such justification:

Proposition 4.2. *The following are all members of* $\mathrm{Ult}(M,E_\nu)$*:*

- $\mathcal{P}(\bigcup_{\nu'<\nu}\bar{N}_{\nu'})$
- *the extender* $\bar{E}_{\nu'}$*, and the map* $\pi_{\nu''}^{-1}\circ\pi_{\nu'}\colon \mathrm{supp}(\bar{E}_{\nu'})\to\mathrm{supp}(\bar{E}_{\nu''})$*, for each* $\nu'<\nu''<\nu$
- *the direct limit of the set* $\{\, N_{\nu'} \mid \nu'<\nu''<\nu \,\}$ *along the maps* $\pi_{\nu''}^{-1}\circ\pi_{\nu'}$*, as well as the injection maps from* $N_{\nu'}$ *into this direct limit*

$\square$

Since $\mathrm{Ult}(M,E_\nu) = \mathrm{Ult}(M,\bar{E}_\nu)$, this proposition allows us to regard the direct limit as a code inside M for the extender E_ν together with its system of subextenders $E_{\nu'}$ for $\nu'<\nu$.

The hypothesis of Theorem 1.4 is more than sufficient to find a mouse M and sequence $\vec{N}$ of submodels satisfying Definition 4.1: this can be done by first defining models M' and $\langle\, N'_\nu \mid \nu<\omega_1 \,\rangle$ satisfying all of the conditions except Clause (7), and then taking M to be the transitive collapse of $\bigcup_{\nu<\omega_1} N'_\nu$. The conditions on M are, in turn, much stronger than is needed to carry out the construction. In view of the fact that there is no clear reason to believe that the actual strength needed is greater that $o(E)=\kappa^{+(\omega+1)}$, it does not seem that there is presently any need to complicate the argument in order to obtain an upper bound closer to $o(E)=\kappa^{+\omega_1}$.

We are now ready to begin the proof of Lemma 3.11. Following Gitik we define, in two subsections, a Prikry type forcing $P(\vec{F})$ depending on a sequence $\vec{F}$ of extenders. Subsections 4.3 and 4.4 develop its properties, and subsection 4.5 describes an equivalence relation $\leftrightarrow$ on its set of conditions. Subsection 4.6 constructs an M_B-generic subset of $i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$, and

Lemma 3.11 under the additional assumption that $\kappa\in B$. Finally subsection 4.8 completes the proof and indicates the technique for proving Theorem 3.6.

4.1 The forcing $P(\vec{F})$

Throughout the definition of the forcing, until the end of subsection 4.6 we work entirely inside the mouse M; in particular all cardinal calculations are carried out inside M. We are interested in defining $P(\vec{E}\restriction\zeta)$, but for the purposes of the recursion used in the definition we allow $\vec{F}$ to be any suitable sequence of extenders. We will not give a definition of the notion of a *suitable sequence* of extenders. All the sequences used in this section are suitable: specifically, all of the sequences $\vec{E}\restriction\delta$ for $\delta < \omega_1$ are suitable, all of the ultrafilters $\{ X \subseteq V_\kappa \mid \vec{E}\restriction\delta \in i^E(X) \}$ concentrate on suitable sequences, and furthermore, if $\vec{F}$ is suitable then so is any $\vec{F}\restriction[\gamma_0, \tau)$ for any $0 \leqslant \gamma_0 \leqslant \tau \leqslant \zeta$.

A generic extension of M by $P(\vec{F})$ would have the form

$$M[G] = M[\vec{\kappa}, \vec{h}],$$

where $\vec{\kappa} = \langle \bar{\kappa}_\gamma \mid \gamma \leqslant \zeta \rangle$ is a closed subset of $\kappa + 1$ with $\bar{\kappa}_\zeta = \kappa$, and $\vec{h} = \langle h_{\nu,\nu'} \mid \zeta \geqslant \nu > \nu' \rangle$ is a sequence of functions $h_{\nu,\nu'} \colon [\bar{\kappa}_\nu, \bar{\kappa}_\nu^+) \to \bar{\kappa}_\nu$. Each of the functions $h_{\nu,\nu'}$ is, individually, Cohen generic; however $h_{\nu,\nu'}$ will be defined, in part, by Prikry type forcing so that some of its values, lying in the interval $[\bar{\kappa}_{\nu'}, \bar{\kappa}_{\nu'}^{+\omega_1})$ form, together with values of other members of the sequence $\vec{h}$, Prikry sequences.

The ordinal $\bar{\kappa}_\nu$ will be, for $\nu < \zeta$, the νth of the principle indiscernibles generated by the forcing. Thus $\bar{\kappa}_\zeta$ is always equal to κ in the forcing $P(\vec{E}\restriction\zeta)$ in M, and $\bar{\kappa}_\zeta = \Omega$ in the forcing $i_\Omega(P(\vec{E}\restriction\zeta))$. If G is the M_B-generic subset of $i_\Omega(P(\vec{E}\restriction\zeta)/{\leftrightarrow})$ constructed in Subsection 4.6, then in $M_B[G]$ the sequence $\langle \bar{\kappa}_\nu \mid \nu < \zeta \rangle$ will be the increasing enumeration of B.

In the generic extension $M_B[G]$, the functions $h_{\nu,\nu'}$ will collectively encode all countable sequences of generators of the model M_B, as follows: Let $\vec{\beta} = \langle \beta_n \mid n < \omega \rangle$ be any sequence of generators in M_B, with β_n being a generator belonging to the ν_nth member $\bar{\kappa}_{\nu_n}$ of B. Then there will be, in M, a sequence $\langle \xi_n \mid n \in \omega \rangle$ of ordinals in $[\kappa, \kappa^+)$ such that $\beta_n = h_{\zeta,\nu_n}(i_\Omega(\xi_n))$ for each $n \in \omega$. Since $\vec{\nu}$ and $i_\Omega(\vec{\xi})$ are both in M_B, it follows that $\vec{\beta} \in M_B[G]$.

The conditions of $P(\vec{F})$ are functions s with a finite domain such that $\zeta \in \operatorname{domain}(s) \subset \zeta + 1$. The values $s(\tau)$ of s are quadruples of the form

$$s(\tau) = (\bar{\kappa}^{s,\tau}, \vec{F}^{s,\tau}, z^{s,\tau}, \vec{A}^{s,\tau}).$$

The first component specifies the value of $\bar{\kappa}_\tau$, and the second component is a suitable sequence of extenders, $\vec{F}^{s,\tau} = \langle F^{s,\tau}_\nu \mid \gamma_0 \leqslant \nu < \tau \rangle$ of extenders, where $\gamma_0 = \max(\{-1\} \cup (\operatorname{domain}(s) \cap \bar{\kappa}^{s,\tau})) + 1$. Neither of these two components will change in conditions $s' \leqslant s$, except that if $\operatorname{domain}(s') \supsetneq \operatorname{domain}(s)$, then $\vec{F}^{s',\tau}$ will be truncated to $\vec{F}^{s,\tau}\restriction[\gamma_0^{s',\tau}, \tau)$.

Like Magidor's forcing in [Mag78], the forcing $P(\vec{F})$ can be factored below any condition s: if $\langle \tau_i \mid i \leqslant n \rangle$ enumerates the domain of s then the forcing $P(\vec{F})\|_s$ of conditions $s' \leqslant s$ in $P(\vec{F})$ is forcing equivalent to $\prod_{i \leqslant n} P(\vec{F}^{s,\tau_i})$.

The third component $z^{s,\tau}$ of $s(\tau)$ ultimately determines the values of the functions $h_{\tau,\nu}$ for $\nu < \tau$. A specific description of this component will be given next, and that will be followed by a specific description of the final component $\vec{A}^{s,\tau}$, which is a sequence $\langle A^{s,\tau}_\nu \mid \gamma_0 \leqslant \nu < \tau \rangle$ of sets $A^{s,\tau}_\nu \in U^{s,\tau}_\nu$ where $U^{s,\tau}_\nu$ is an ultrafilter which will be derived from $F^{s,\tau}_\tau$. As in other Prikry type forcings, the set $A^{s,\tau}_\nu$ is used to limit the possible extensions $s' \leqslant s$ with $\nu \in \operatorname{domain}(s')$.

	0	$\cdots$	$\gamma_0 - 1$	γ_0	$\cdots$	γ	$\cdots$
τ	$f^z_{\tau,0}$	$\cdots$	f^z_{τ,γ_0-1}	$(a^z_{\tau,\gamma_0}, f^z_{\tau,\gamma_0})$	$\cdots$	$(a^z_{\tau,\gamma}, f^z_{\tau,\gamma})$	$\cdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	
γ	$f^z_{\gamma,0}$	$\cdots$	f^z_{γ,γ_0-1}	$(a^z_{\tau,\gamma_0}, f^z_{\gamma,\gamma_0})$	$\cdots$		
$\vdots$	$\vdots$		$\vdots$	$\vdots$			
$\gamma_0 + 1$	$f^z_{\gamma_0+1,0}$	$\cdots$	$f^z_{\gamma_0+1,\gamma_0-1}$	$(a^z_{\gamma_0+1,\gamma_0}, f^z_{\gamma_0+1,\gamma_0})$			
γ_0	$f^z_{\gamma_0,0}$	$\cdots$	$f^z_{\gamma_0,\gamma_0-1}$				

Figure 1: The middle component $z^{s,\tau}$ of $s(\tau)$. The element at row α and column β is used to determine $h_{\alpha,\beta}$. In the case of the top row, this determination is direct; for the other rows this is indirect, via their use in defining the ultrafilters $U^{s,\tau}_\gamma$ from which the sets $A^{s,\tau}_\alpha$ are taken.

The tableau $z^{s,\tau}$. The third component $z^{s,\tau}$ of $s(\tau)$ is a tableau having the form represented in Figure 1. It contains, for each pair (γ, ν) of ordinals with $\tau \geqslant \gamma \geqslant \gamma_0 > \nu \geqslant 0$, a function $f^z_{\gamma,\nu}$, and for each pair (γ, ν) with $\tau \geqslant \gamma > \nu \geqslant \gamma_0$, a pair of functions $(a^z_{\gamma,\nu}, f^z_{\gamma,\nu})$. The function $f^z_{\gamma,\nu}$ or pair of functions $(a^z_{\gamma,\nu}, f^z_{\gamma,\nu})$ will ultimately be used to determine the values of the Cohen function $h_{\gamma,\nu}$. The elements in the first row will be used to directly determine $h_{\tau,\nu}$; while elements in a row labeled $\gamma < \tau$ will be used indirectly to determine $h_{\gamma,\nu}$ by being used in the definition of the ultrafilter $U^{s,\tau}_\gamma$.

The domain of the function $f^z_{\gamma,\nu}$ is contained in the interval $[\bar\kappa^z_\tau, (\bar\kappa^z_\tau)^+)$ and is of size at most $\bar\kappa^z_\tau$. This is a standard Cohen condition for a function from $(\bar\kappa^z_\tau)^+$ into $\bar\kappa^z_\tau$, except that it takes values of two different forms:

1. $f^z_{\gamma,\nu}(\xi) = \xi' \in \bar\kappa^z_\tau$, and
2. $f^z_{\gamma,\nu}(\xi) = h_{\gamma',\nu}(\xi')$ for some γ' in the interval $\gamma > \gamma' > \nu$ and $\xi' \in \bar\kappa^z_\tau$

The first is the usual form for a Cohen condition and asserts that $h_{\gamma\nu}(\xi) = \zeta'$. More specifically, if s is a condition with $z^{s,\tau} = z$ and $f^z_{\tau,\gamma}(\xi) = \xi'$, then $s \Vdash \dot h_{\tau,\gamma}(\xi) = \xi'$. In the second form, the value $h_{\gamma',\nu}(\xi')$, of $f(\xi)$ may be taken as a formal expression. In this case the value of the name $\dot h(\xi)$ is defined by

$$\begin{aligned} &\text{if } s \Vdash \dot h_{\tau,\nu}(\xi') = \xi'' \text{ then} && s \Vdash \dot h_{\tau,\nu}(\xi) = \xi'', \\ &\text{if } s \Vdash \xi' \notin \operatorname{domain}(\dot h_{\gamma',\nu}) \text{ then} && s \Vdash \dot h_{\tau,\nu}(\xi) = 0, \\ &\text{and otherwise} && s \nVdash \dot h_{\tau,\nu}(\xi). \end{aligned}$$

This definition uses recursion on τ, since $s \Vdash \dot h_{\gamma',\nu}(\xi') = \xi''$ depends only on $s \restriction \gamma' + 1$. In the first of these three cases, $s \Vdash \dot h_{\gamma',\nu}(\xi') = \xi''$, we will regard the forms $f^z_{\tau,\nu}(\xi) = \xi''$ and $f^z_{\tau,\nu}(\xi) = h_{\gamma',\nu}(\xi')$ as being identical.

The domain of the function $a^{s,\tau}_{\gamma,\nu}$ is also a subset of the interval $[\bar\kappa_\tau, \bar\kappa_\tau^+)$ of size at most $\bar\kappa_\tau$, but it is disjoint from the domain of $f^{s,\tau}_{\gamma,\nu}$. The range of $a^{s,\tau}_{\gamma,\nu}$ is a subset of $\operatorname{supp}(F^{s,\tau}_\nu)$. The intention is that if $a^{s,\tau}_{\tau,\nu}(\xi) = \alpha$, then $h_{\tau,\nu}(\xi)$ will be a Prikry indiscernible for the ultrafilter $(F^{s,\tau}_\nu)_\alpha = \{\, x \in \mathcal{P}(\bar\kappa) \mid \alpha \in i^{F^{s,\tau}_\nu}(x) \,\}$.

Although we usually regard $a^{s,\tau}_{\gamma,\nu}$ as simply a function, it requires a moderately complicated bookkeeping structure.

Definition 4.3. We put the following requirements on these functions:

1. The domains of $a^{s,\tau}_{\gamma,\nu}$ and $f^{s,\tau}_{\gamma,\nu}$ are disjoint.

2. The domain of $a^{s,\tau}_{\gamma,\nu}$ is equipped with a layering, which we write as $a^{z}_{\gamma,\nu} = \bigcup_{\lambda<\kappa_\gamma} a^{z}_{\gamma,\nu}|\lambda$. This layering satisfies (i) $\left|(a^{z}_{\gamma,\nu}|\lambda)\right| \leqslant \lambda$ for each cardinal $\lambda < \kappa_z$ such that s does not force $\bar\kappa_\nu \neq \lambda$, (ii) $a^{z}_{\gamma,\nu}|\lambda \subseteq a^{z}_{\gamma,\nu}|\lambda'$ whenever $\lambda < \lambda'$, and (iii) $a^{z}_{\gamma,\nu}|\lambda = \bigcup_{\lambda'<\lambda} a^{z}_{\gamma}|\lambda'$ for limit cardinals $\lambda < \bar\kappa_\tau$.

3. If $\tau \geqslant \gamma > \gamma' > \nu$ then $a^{s,\tau}_{\gamma,\nu} \subseteq a^{s,\tau}_{\gamma',\nu}$.

For ease of referring to this structure, we will use the following definition:

Definition 4.4. If s is a condition, with $\tau \in \operatorname{domain}(s)$, then the *pattern* of a function $a^{s,\tau}_{\gamma,\nu}$ in $z^{s,t}$ comprises the following elements: (i) the domain of $a^{s,\tau}_{\gamma,\nu}$, (ii) the layering of its domain, and (iii) the preordering $\preccurlyeq$ on the domain of $a^{s,\tau}_{\gamma,\nu}$ defined by $\xi \preccurlyeq \xi'$ if $a^{s,\tau}_{\gamma,\nu}(\xi) \leqslant a^{s,\tau}_{\gamma,\nu}(\xi')$.

The pattern of a column $\langle\, a^{s,\tau}_{\gamma,\nu} \mid \tau \geqslant \gamma > \nu \,\rangle$ from the tableau comprises the pattern of each member of the column.

We will occasionally speak of the pattern of a condition s, or of other related objects. This will mean the collection containing the pattern of each column in the associated tableau or tableaux.

This completes the definition of the tableau $z^{s,\tau}$. Before continuing with the definition of the sequence $\vec{A}^{s,\tau}$, we make some some general remarks on the reasons for the design of the tableau, and the design of the forcing in general. An important aim is to ensure that the forcing produces Prikry indiscernibles for the ultrafilters $(F^{s,\tau}_\nu)_\alpha$ of the extender on $\bar\kappa_\tau$, but that no information about the association between these indiscernibles and the ordinal α is included in the model $M_B[G]$. The forcing involves three techniques to arrange this, two of which have already been touched on: (i) The mixture of Cohen forcing and Prikry forcing gives a background in which in which the Prikry forcing is hidden, so that there is no way to distinguish values $h_{\delta,\nu}(\xi)$ coming from Prikry forcing from those which come from Cohen forcing. (ii) The use of the function $a^{s,\tau}_{\gamma,\nu}(\xi)$ in the condition, instead of its value α, ties the Prikry indiscernible to the essentially arbitrary ordinal $\xi \in [\bar\kappa^{s,\tau}_\gamma, (\bar\kappa^{s,\tau}_\gamma)^+)$, thus avoiding any explicit tie between the indiscernible and the ordinal α for which it is a Prikry indiscernible. (iii) The equivalence relation $\leftrightarrow$ on $P(\vec{F})$, which will be defined in Section 4.5, will explicitly disassociate the Prikry indiscernible from the ordinal α.

To see why this disassociation is necessary, recall that the observation that $M_B[G]$ is closed under countable sequences of ordinals used the claim that for every generator $\beta = i_{\nu'}(\bar\beta')$ belonging to some $\bar\kappa_\nu = \kappa_{\nu'} \in B$, there is some $\xi \in [\kappa, \kappa^{+\omega_1})$ such that $\beta = h_{\zeta,\nu}(i_\Omega(\xi))$. If we used generic subset G of the unmodified forcing $P(\vec E\restriction\zeta)$, rather than generating a generic subset of $P(\vec E\restriction\zeta)/\leftrightarrow$, then the only generators of κ_γ for which we would have names $h_{\zeta,\gamma}(i_\Omega(\xi))$ would be those in $\bigcup_{\nu<\zeta} i_\gamma[\operatorname{supp}(E_\nu)]$. By enforcing the disassociation, we make it possible to modify the conditions to give such names to all generators.

The sets $A^{s,\tau}_\gamma$. We have finished the definition of the conditions $s \in P(\vec F)$ except for characterization of the sets $A^{s,\tau}_\gamma$ and the ultrafilters $U^{s,\tau}_\gamma$ of which they are members.

For convenience, we write P^*_τ for the set of quadruples which are possible values of $s(\tau)$. In Section 4.2 we will define an order $\leqslant^*$ on the sets P^*_τ which will induce the direct order $\leqslant^*$ on $P(\vec F)$. In the rest of this section we assume, as a recursion hypothesis, that this order has already been defined on all P^*_γ for $\gamma < \tau$.

The members of $A^{s,\tau}_\gamma$ are members of P^*_γ with some additional information. Specifically, $A^{s,\tau}_\gamma \subseteq P^*_{\gamma,\tau}$ where

Definition 4.5. The members of $P^*_{\gamma,\tau}$ are quadruples

$$w = (\bar\kappa^w_\gamma, \vec F^w, z^w, \vec A^w)$$

satisfying the following conditions:

1. $\bar\kappa^w_\gamma$ and $\vec F^w$ are as described above for P^*_γ.
2. z^w has the form of the tableau in Figure 2.
3. If $\tau \geqslant \nu \geqslant \gamma + 1 > \nu' \geqslant \gamma_0$, then $a^{z^w}_{\nu,\nu'}$ is a function with a domain which has size $\bar\kappa^w$ and is contained in $[\bar\kappa^w, (\bar\kappa^w)^+)$, and with range contained in $\Omega\setminus\operatorname{supp}(F_\gamma)$.
4. $(\bar\kappa^w_\gamma, \vec F^w, z^w\restriction[\gamma_0,\gamma), \vec A^w) \in P^*_\gamma$, where $z^w\restriction[\gamma_0,\gamma)$ is the restriction of z to the rows with indices in the interval $[\gamma_0,\gamma)$, that is, those below the line in Figure 2.

If $w', w \in P^*_{\gamma,\tau}$ then $w' \leqslant^* w$ if

1. $(\bar\kappa^{w'}_\gamma, \vec F^{w'}, z^{w'}\restriction[\gamma_0,\gamma), \vec A^{w'}) \leqslant^* (\bar\kappa^w_\gamma, \vec F^w, z^w\restriction[\gamma_0,\gamma), \vec A^w)$ in P^*_γ, and
2. If $\tau \geqslant \nu \geqslant \gamma + 1 > \nu' \geqslant \gamma_0$, then $a^{z^{w'}}_{\nu,\nu'} \supseteq a^{z^w}_{\nu,\nu'}$.

We are now ready to finish the finish the definition of the sets P^*_τ, and hence of the set of conditions of the forcing $P(\vec F)$, by defining ultrafilters $U^{s,\tau}_\gamma$ on subsets of $P^*_{\gamma,\tau}$:

Definition 4.6. $U^{s,\tau}_\gamma$ is the ultrafilter on subsets of $P^*_{\gamma,\tau}$ defined by

$$x \in U^{s,\tau}_\gamma \iff s(\tau)\uparrow\gamma \in i^{F_\gamma}(x). \tag{1}$$

$$
\begin{array}{c|cccccc}
 & 0 & \cdots & \gamma_0 - 1 & \gamma_0 & \cdots & \gamma \\
\hline
\tau & & & & a^z_{\tau,\gamma_0} & \cdots & a^z_{\tau,\gamma} \\
\vdots & & & & \vdots & & \vdots \\
\gamma + 1 & & & & a^z_{\gamma+1,\gamma_0} & \cdots & a^z_{\gamma+1,\gamma} \\
\hline
\gamma & f^z_{\gamma,0} & \cdots & f^z_{\gamma,\gamma_0-1} & (a^z_{\gamma,\gamma_0}, f^z_{\gamma,\gamma_0}) & \cdots & \\
\vdots & \vdots & & \vdots & \vdots & & \\
\gamma_0 + 1 & f^z_{\gamma_0+1,0} & \cdots & f^z_{\gamma_0+1,\gamma_0-1} & (a^z_{\gamma_0+1,\gamma_0}, f^z_{1,\gamma_0}) & & \\
\gamma_0 & f^z_{\gamma_0,0} & \cdots & f^z_{\gamma_0,\gamma_0-1} & & &
\end{array}
$$

Figure 2: The third component z^w of a member of $A^{s,\tau}_\gamma$. The entry in row α and column β is used in the determination of $h_{\alpha,\beta}$.

Here $s(\tau)\uparrow\gamma$ is a member of $P^*_{\gamma,\tau}$ obtained by restricting $s(\tau) \in P^*_\tau$ as follows:

$$s(\tau)\uparrow\gamma = (\bar{\kappa}^{s,\tau}, \vec{F}^{s,\tau} \restriction \gamma, (z^{s,\tau})^{*\gamma}, \vec{A}^{s,\tau}\uparrow\gamma), \tag{2}$$

where

1. $(z^{s,\tau})^{*\gamma}$ is a tableau as in Figure 2 which is obtained from $z^{s,\tau}$ by deleting all columns with index larger than γ, and retaining only the functions $a^z_{\nu,\nu'}$ from the rows with index $\nu > \gamma$, and
2. $\vec{A}^{s,\tau}\uparrow\gamma = \langle \{ w\restriction\gamma \mid w \in A^{s,\tau}_{\gamma'} \} \mid \gamma' < \gamma \rangle$, $w\restriction\gamma = (\bar{\kappa}^w, \vec{F}^w, z^w \restriction [\gamma_0, \gamma), \vec{A}^w)$.

Definition 4.7. The set P^*_τ is the set of quadruples $t = (\bar{\kappa}^t, \vec{F}^t, z^t, \vec{A}^t)$ satisfying the following conditions:

1. $\bar{\kappa}^t$, $\vec{F}^t$ and z^t are as specified above.
2. $\vec{A}^t = \langle A^t_\gamma \mid \gamma_0 \leqslant \gamma < \tau \rangle$, where $A^t_\gamma \in U^t_\gamma$.
3. If $w \in A^t_\gamma$ and $\gamma' < \gamma$, then

$$\begin{aligned} A^w_{\gamma'} = \{ w' \restriction [\gamma_0, \gamma) \mid w' \in A^t_{\gamma'} \ \&\ \bar{\kappa}^{w'} < \bar{\kappa}^w \\ \&\ \forall \nu, \nu' \, (\gamma_0 \leqslant \nu \leqslant \gamma < \nu' \leqslant \tau \implies a^{w'}_{\nu',\nu} = a^w_{\nu',\nu}) \}. \end{aligned}$$

We remark that clause (3) of Definition 4.7 is included here only for convenience, as omitting it would give an equivalent forcing. In the next subsection, we will implicitly add more such regularizing conditions when we define add(s, w), as any $w \in A^{s,\tau}_\gamma$ for which add(s, w) is undefined has no effect.

This completes the definition of the set of conditions for the forcing $P(\vec{F})$.

4.2 The partial orders of $P(\vec{F})$.

Since $P(\vec{F})$ is a Prikry type forcing notion, we need to define both a direct extension order $\leqslant^*$ and a forcing order $\leqslant$. We will begin by defining the one-step extension, $\operatorname{add}(s,w) \leqslant s$, which which is the basic extension adding a single new ordinal to the domain of s. We will then define the direct extension $\leqslant^*$. The forcing extension $\leqslant$ is then the smallest transitive relation which contains $\leqslant^*$ such that $\operatorname{add}(s,w) \leqslant s$ whenever $w \in A^{s,\tau}_{\gamma}$ for some $\tau \in \operatorname{domain}(s)$ and $\gamma < \tau$.

The one-step extension The one-step extension in this forcing is corresponds to the extension in Prikry forcing which simply adds one new ordinal to the finite sequence. If $w \in A^{s,\tau}_{\gamma}$ then $\operatorname{add}(s,w)$ is the weakest extension of s which has γ in its domain; that is, if $t \leqslant s$ and $\gamma \in \operatorname{domain}(t)$, then $t \leqslant \operatorname{add}(s,w) < s$ for some $w \in A^{s,\tau}_{\gamma}$. The first definition uses the portion of the tableau of Figure 2 lying above the line to resolve the corresponding functions $a^{s,\tau}_{\nu,\nu'}$ of s to Cohen conditions:

Definition 4.8. Suppose that $w \in A^{s,\tau}_{\gamma}$ and $\tau \geqslant \nu > \gamma \geqslant \nu' \geqslant \gamma_0$; and write $a = a^{s,\tau}_{\nu,\nu'}$ and $a' = a^{w}_{\nu,\nu'}$. Assume that a' has same pattern as $a|\bar{\kappa}^w_{\gamma}$, so in particular $\operatorname{otp}(\operatorname{domain}(a')) = \operatorname{otp}(\operatorname{domain}(a|\bar{\kappa}^w_{\gamma}))$. Then the Cohen condition $f_{a,a'}$ determined by a and a' is defined as follows:

Let $\xi \in \operatorname{domain}(a)$ be arbitrary, let β be such that ξ is the βth member of $\operatorname{domain}(a)$, and if $\xi \in \operatorname{domain}(a|\bar{\kappa}^w_{\gamma})$ then let ξ' be the βth member of $\operatorname{domain}(a')$. Then

$$f_{a,a'}(\xi) = \begin{cases} a'(\xi') & \text{if } \xi \in \operatorname{domain}(a|\bar{\kappa}^w_{\gamma}) \text{ and } \gamma = \nu', \\ h_{\gamma,\nu'}(\xi') & \text{if } \xi \in \operatorname{domain}(a|\bar{\kappa}^w_{\gamma}) \text{ and } \gamma > \nu', \\ 0 & \text{if } \xi \in \operatorname{domain}(a) \setminus \operatorname{domain}(a|\bar{\kappa}^w_{\gamma}). \end{cases} \tag{3}$$

The second case uses the second form of the value of a Cohen condition; because of the requirement that $a^{s,\tau}_{\nu,\nu'} \subseteq a^{s,\tau}_{\gamma,\nu'}$, the ultimate value of $h_{\nu,\nu'}(\xi)$ can still be regarded as a Prikry indiscernible for α. This corresponds to Magidor's generalization of Prikry forcing in [Mag78].

Now we are ready define the *one-step extension* $s' = \operatorname{add}(s,w)$.

Definition 4.9. Suppose that $w \in A^{s,\tau}_{\gamma}$ where $\tau = \min(\operatorname{domain}(s) \setminus \gamma)$. Then $s' = \operatorname{add}(s,w)$ is the condition with $\operatorname{domain}(s') = \operatorname{domain}(s) \cup \{\gamma\}$ defined as follows:

First, $s' \upharpoonright \operatorname{domain}(s) \setminus \{\tau,\gamma\} = s \upharpoonright \operatorname{domain}(s) \setminus \{\tau\}$; thus, only $s'(\gamma)$ and $s'(\tau)$ remain to be defined. As before, let $\gamma_0 = \max(\operatorname{domain}(s) \cap \tau) + 1$, or $\gamma_0 = 0$ if γ is the least member of $\operatorname{domain}(s)$. Fix $w = (\bar{\kappa}^w, \vec{F}^w, z^w, \vec{A}^w) \in A^{s,\tau}_{\gamma}$. The value $s'(\gamma)$ is specified by w:

$$s'(\gamma) = (\bar{\kappa}^w_{\gamma}, \vec{F}^w, z^w \upharpoonright [\gamma_0, \gamma], \vec{A}^w).$$

Finally, the definition of $s'(\tau)$ is by recursion over the pairs (τ,γ): We set $s'(\tau) = (\bar{\kappa}^{s'}, \vec{F}^{\tau,s'}, z^{s',\tau}, \vec{A}^{s',\tau})$, where

1. $\bar{\kappa}^{s'}_{\tau} = \bar{\kappa}^{s}_{\tau}$ and $\vec{F}^{s',\tau} = \vec{F}^{s,\tau}\restriction[\gamma+1,\tau)$,

2. $z^{s',\tau}$ is obtained from $z^{s,\tau}\restriction(\gamma,\tau]$ by setting $f^{s',\tau}_{\nu,\nu'} = f^{s,\tau}_{\nu,\nu'} \cup f_{a,a'}$, using Definition 4.8, whenever $\tau \geqslant \nu > \gamma \geqslant \nu' \geqslant \gamma_0$, and

3. If $\gamma < \nu < \tau$, then

$$A^{s',\tau}_{\nu} = \{\, \sigma(w') \mid w' \in A^{s,\tau}_{\nu} \wedge \bar{\kappa}^{w}_{\gamma} < \bar{\kappa}^{w'}_{\nu} \,\} \tag{4}$$

where the function σ is defined by

$$\sigma(w')\restriction(\gamma,\nu] = \operatorname{add}(w'\restriction(\gamma,\nu], w\!\uparrow\!\nu), \text{ and}$$
$$\sigma(w')\restriction(\nu,\tau] = w'\restriction(\nu,\tau].$$

Clause (3) uses the recursion, and abuses notation by identifying the quadruple $w'\restriction(\gamma,\nu] \in P^*_{\nu}$ with the condition $\{(\nu, w'\restriction(\gamma,\nu])\} \in P(\vec{F}^{w'}\restriction(\gamma,\nu])$ having domain $\{\nu\}$.

If any part of the definition of $\operatorname{add}(s,w)$ cannot be carried out as described, then $\operatorname{add}(s,w)$ is left undefined. Note that the set of w for which it is defined is a member of $U^{s,\tau}_{\gamma}$, so that we can assume without loss of generality that $\operatorname{add}(s,w)$ is defined for every $w \in A^{s,\tau}_{\gamma}$.

The direct extension order. This completes the definition of the one-step extension, and we complete the definition of the forcing $P(\vec{F})$ by defining the direct extension ordering, $\leqslant^*$. This is just the cartesian product of orderings defined on the sets P_{γ}:

Definition 4.10. We will define an order $\leqslant^*$ on P^*_{τ}; the order on $P(\vec{F})$ is then defined by

$$s' \leqslant^* s \iff \operatorname{domain}(s') = \operatorname{domain}(s) \wedge \forall\gamma \in \operatorname{domain}(s)\ s'(\gamma) \leqslant^* s(\gamma).$$

The definition of the order $\leqslant^*$ on P^*_{τ} uses recursion on τ: we assume that the relation $\leqslant^*$ on $P^*_{\tau'}$ has been defined for all $\tau' < \tau$. Then for any $t' = (\bar{\kappa}^{t'}, \vec{F}^{t'}, z^{t'}, \vec{A}^{t'})$ and $t = (\bar{\kappa}^{t}, \vec{F}^{t}, z^{t}, \vec{A}^{t})$ in $P^*(\tau)$ we say that $t' \leqslant^* t$ if the following conditions are satisfied.

1. $\bar{\kappa}^{t'} = \bar{\kappa}^{t}$ and $\vec{F}^{t'} = \vec{F}^{t}$.

2. $a^{t'}_{\gamma,\gamma'}\restriction\operatorname{domain}(a^{t}_{\gamma,\gamma'}) = a^{t}_{\gamma,\gamma'}$ for each pair (γ,γ') for which they are defined, and the induced pattern on $\{\, a^{t'}_{\gamma,\gamma'}\restriction\operatorname{domain}(a^{t}_{\gamma,\gamma'}) \mid \tau \geqslant \gamma > \gamma' > \gamma_0 \,\}$ is the same as that of $\{\, a^{s,\tau}_{\gamma,\gamma'} \mid \tau \geqslant \gamma > \gamma' > \gamma_0 \,\}$.

3. For each $\gamma \in [\gamma_0,\tau)$ and each $w' \in A^{t'}_{\gamma}$ there is $w \in A^{t}_{\gamma}$ such that

   (a) $w'\restriction[\gamma_0,\gamma] \leqslant^* w\restriction[\gamma_0,\gamma]$ in P^*_{γ}.

   (b) $a^{w'}_{\nu,\nu'} \supseteq a^{w}_{\nu,\nu'}$ for all pairs (ν,ν') such that $\tau \geqslant \nu > \gamma \geqslant \nu' \geqslant \gamma_0$, with the pattern being preserved as in clause (2) above.

(c) For all pairs (ν, ν') with $\tau \geqslant \nu > \nu' \geqslant \gamma_0$, if f' and f are the functions induced, using Definition 4.8, by the pair $(a^{t'}_{\nu,\nu'}, a^{w'}_{\nu,\nu'})$ and the pair $(a^{t}_{\nu,\nu'}, a^{w,}_{\nu,\nu'})$ respectively, then $f = f' \upharpoonright \mathrm{domain}(a^t_{\nu,\nu'})$.

4. $f^{t'}_{\nu,\nu'} \upharpoonright \mathrm{domain}(f^t_{\nu,\nu'}) = f^t_{\nu,\nu'}$ for each pair ν, ν' for which they are defined.

The clauses (3b) and (3c) assert that $\mathrm{add}(s', w')(\tau) \leqslant^* \mathrm{add}(s, w)(\tau)$. Clause (3) adapts the requirement $A^{s'} \subseteq A^s$ from Prikry forcing to deal with he complication that $U^{s,\tau}_\gamma \neq U^{s',\tau}_\gamma$: it extends the ordering $\leqslant^*$ on P^*_γ to $P^*_{\gamma,\tau}$ and then asserts that $A^{s',\tau}_\gamma \subseteq \{\, w' \mid \exists w \in A^{s,\tau}_\gamma \; w' \leqslant^* w \,\}$.

In Gitik's forcing, this corresponds to his use, in the definition of the direct order, of a predetermined set of witnesses $\pi_{\alpha^{s'},\alpha^s}$ to the fact that $U_{\alpha_{s'}} \leqslant_{\mathrm{RK}} U_{s_\alpha}$ where the ultrafilters U_α come from a predetermined sequence of ultrafilters.

This completes the definition of the forcing $(P(\vec{F}), \leqslant^*, \leqslant)$.

4.3 Properties of the forcing $P(\vec{F})$

Definition 4.11. If $\vec{w}$ is a sequence of length n, then we write $\mathrm{add}(s, \vec{w})$ for the condition defined by recursion as $\mathrm{add}(s, \vec{w}) = s$ if $n = 0$, and $\mathrm{add}(s, \vec{w}) = \mathrm{add}(\mathrm{add}(s, \vec{w} \upharpoonright (n-1)), w_{n-1})$ if $n > 0$.

Proposition 4.12. *Suppose that $s \leqslant t$. Then there is $\vec{w}$ such that $s \leqslant^* \mathrm{add}(t, \vec{w}) \leqslant t$*

Proof. The proposition will follow by induction on $|\,\mathrm{domain}(t) \setminus \mathrm{domain}(s)|$ once we show that for any t' and $w' \in A^{t'}_\gamma$ such that $s = \mathrm{add}(t', w') \leqslant t' \leqslant^* t$, there is some $w \in A^t_\gamma$ such that $s \leqslant^* \mathrm{add}(t, w) < t$. Let $w \in A^t_\gamma$ be as given by Clause 3 of the Definition 4.10. Then $s \leqslant^* \mathrm{add}(t, w)$, for by Clause 3a, $s(\gamma) \leqslant^* \mathrm{add}(t, w)(\gamma)$, by clauses 3b,c and 4 we have $s(\tau) \leqslant^* \mathrm{add}(s, \bar{w})(\tau)$, and for all $\gamma' \in \mathrm{domain}(s') \setminus \{\tau, \gamma\}$ we have $t(\gamma') \leqslant^* s'(\gamma') = s'(\gamma')$. $\square$

Proposition 4.13. *Suppose $t \leqslant s$ and $\gamma \in \mathrm{domain}(t) \setminus \mathrm{domain}(s)$, and let $\tau = \min(\mathrm{domain}(s) \setminus \gamma)$. Then there is $w \in A^{s,\tau}_\gamma$ such that $t \leqslant \mathrm{add}(s, w) < s$.*

Proof. By Proposition 4.12, there is a sequence $\vec{w}$ so that $t \leqslant^* \mathrm{add}(s, \vec{w}) \leqslant s$. Thus it only remains to show that the order of the extension $\mathrm{add}(s, \vec{w})$ can be permuted, that is, that there is $\vec{w}'$ such that $\mathrm{add}(s, \vec{w}) = \mathrm{add}(s, \vec{w}')$ and $w'_0 \in A^{s,\tau}_\gamma$, in which case $w = w'_0$ satisfies the proposition. This will follow by an easy induction once we show that the order of two consecutive one-step extensions can be reversed.

Suppose then that $t = \mathrm{add}(\mathrm{add}(s, w), v)$, with $w \in A^{s,\tau}_{\gamma_w}$ adding γ_w to the domain of s, and v then adding γ_v to the domain of $\mathrm{add}(s, w)$. We want to show that there are w' and v', adding γ_w and γ_v respectively, so that $t = \mathrm{add}(\mathrm{add}(s, v'), w')$. We can assume that γ_w and γ_v are not separated by a member of $\mathrm{domain}(s)$, that is, (using the notation from Definition 4.9) $\gamma_0 \leqslant \gamma_w, \gamma_v < \tau$, for otherwise we have $\mathrm{add}(\mathrm{add}(s, w), v) = \mathrm{add}(\mathrm{add}(s, v), w)$.

This is the purpose of including Clause (3) of Definition 4.7. In the case that $\gamma^w < \gamma^v$, we take w' as in that clause, and $v' = \sigma(v)$ where the function σ is from Clause (3) of the Definition 4.9 of $\operatorname{add}(s, w)$; conversely, in the case that $\gamma^v < \gamma^w$, we take v' so that $v = \sigma(v')$ and $w' = u \in A^{s,\tau}_{\gamma_w}$ so that $w = u'$. $\square$

Proposition 4.14. *1. If $t \in (P^*_\gamma, \leqslant^*)$, then $(P^*_\gamma\|_t, \leqslant^*)$ is $<\bar\kappa^t$-closed.*

2. The order $(P(\vec F), \leqslant^*)$ is countably closed.

3. If $s \in P(\vec F)$ with $\gamma = \min(\operatorname{domain}(s))$ then $(P(\vec F)\|_s, \leqslant^*)$ is $\bar\kappa^{s,\gamma}$-closed.

4. If $s \in P(\vec F)$ and $\gamma \in \operatorname{domain}(s)$ with $\gamma < \zeta$ then

$$P(\vec F)\|_s \cong P(\vec F^{s,\gamma})\|_{s\restriction(\gamma+1)} \times R$$

where $(R, \leqslant^*_R)$ is $(2^{|P(\vec F^{s,\gamma})|})^+$-closed.

Here we write $P(\vec F)\|_s$ for $\{\, s' \in P(\vec F) \mid s' \leqslant s \,\}$.

Proof. Clauses 2 and 3 will follow immediately from Clause 1, and Clause 4 follows from Clause 1 together with Proposition 4.13. The proof of Clause 1 is by induction on $\operatorname{length}(\vec F)$: suppose that $\langle w_\nu \mid \nu < \theta \rangle$ is a $\leqslant^*$-descending sequence of length $\theta < \bar\kappa^{w_0}$ in $P^*(\vec F)$. We want to show that there is an infinum $w_\theta = \bigwedge_{\nu<\theta} w_\nu$ of this sequence in $P^*(\vec F)$. The only problematic element of the definition of w_θ is the definition of the sets $A^{w_\theta}_\gamma$. We set

$$\begin{aligned} v \in A^{w_\theta}_\gamma \iff \bar\kappa^v > \theta \;\&\; \exists \vec v = \langle v_\nu \mid \nu < \gamma \rangle \, (v = \bigwedge_{\nu<\theta} v_\nu \\ \&\; \forall \nu < \gamma \, v_\nu \in A^{w_\nu}_\gamma \;\&\; \forall (\nu, \nu') \, (\nu' < \nu < \gamma \implies w_{\nu'} \leqslant^* w_\nu)). \end{aligned}$$

To see that this works, we need to verify that $A^{w_\theta}_\gamma \in U^{w_\theta}_\gamma$ for each $\gamma < \operatorname{length}(\vec F)$. But this is the induction hypothesis: we have $w^{*\gamma}_\theta = \bigwedge_{\nu<\theta} w^{*\gamma}_\nu$ for each $\gamma < \operatorname{length}(\vec F)$. $\square$

The factorization asserted in Clause(4) is a general fact about Prikry-type forcings in the line of Magidor's [Mag78]. This, together with the observation that The forcing R in Clause (4) is the product of $P(\vec F\restriction(\gamma,\zeta))\|_{s\restriction[\gamma+1,\zeta]}$ with the forcing order for adding additional Cohen subsets of cardinals larger than $\bar\kappa_\gamma$, is frequently useful: In order to prove a property of an arbitrary condition $s \in P(\vec F)$ it is sufficient to prove it for conditions whose domain is a singleton, provided that it holds of the Cohen forcing and is preserved under finite products.

The next lemma extends Lemma 4.14 to allow diagonal intersections of length $\bar\kappa^t$:

Clause (4) will frequently allow us to simplify a proof by considering only conditions s with $\operatorname{domain}(s) = \{\zeta\}$: suppose we are trying to prove a property of s and $P(\vec F)\|_s$, and the property is true of $\bar\kappa_\zeta$-closed forcing and is preserved under products. If we can show that the property holds for the case $\operatorname{domain}(s) = \{\zeta\}$ then it follows by induction that it is true in general: Apply Clause (4) to write $P(\vec{F}) = P(\vec{F}^{s,\gamma})\|_{s\upharpoonright(\gamma+1)} \times R$. Then by our assumptions the property holds of R, but by the induction hypothesis it also holds of $P(\vec{F}^{s,\gamma})\|_{s\upharpoonright(\gamma+1)}$ and $s\upharpoonright\gamma+1$, so it follows that it holds of s and $P(\vec{F})\|_s$.

Lemma 4.15. *Suppose that s is a condition in $P(\vec{F})$, $\gamma < \tau = \min(\mathrm{domain}(s))$, and that D is a subset of $P(\vec{F})$ which is open dense in $(P(\vec{F}), \leqslant^*)$ below $\mathrm{add}(s,w)$ for all $w \in A^{s,\tau}_\gamma$. Then there is a $s' \leqslant^* s$ such that $s'' \in D$ for all $s'' \leqslant s'$ with $\gamma \in \mathrm{domain}(s'')$.*

Proof. By Proposition 4.13 it will be enough to show that there is $s' \leqslant^* s$ such that $\mathrm{add}(s',w) \in D$ for all $w \in A^{s',\tau}_\gamma$, and by Proposition 4.14(4) we can assume that $\tau = \sup(\mathrm{domain}(s))$.

We will construct s' in two steps. The first step will find $s_{\bar\kappa_\tau} \leqslant^* s$ and a function $\sigma\colon A^{s,\tau}_\gamma \to P^*_\gamma$ such that $\mathrm{add}(s_{\bar\kappa_\tau}, \sigma(w)) \in D$ for each $w \in A^{s,\tau}_\gamma$, with an abuse of notation since $\sigma(w) \notin A^{s_{\bar\kappa_\tau},\tau}_\gamma$. The second step will use the function σ to modify $s_{\bar\kappa_\tau}$ to $s' \leqslant^* s$ with $A^{s',\tau}_\gamma = \{\, \sigma(w) \mid w \in A^{s,\tau}_\gamma \,\}$ such that $\mathrm{add}(s', \sigma(w)) = \mathrm{add}(s_{\bar\kappa_\tau}, \sigma(w))$ for all $w \in A^{s,\tau}_\gamma$.

Enumerate $A^{s,\tau}_\gamma$ as $\{\, w_\nu \mid \nu < \bar\kappa^s \,\}$ so that $\nu' \leqslant \nu$ implies $\bar\kappa^{w_{\nu'}}_\gamma \leqslant \bar\kappa^{w_\nu}_\gamma$. We will define a $\leqslant^*$-decreasing sequence of conditions $\langle\, s_\nu \mid \nu \leqslant \kappa \,\rangle$ in R, along with the function σ, having the following properties:

1. $s_0 = s$, and $s_\nu = \bigwedge_{\nu'<\nu} s_{\nu'}$ if $\nu \leqslant \bar\kappa^s_\tau$ is a limit ordinal.
2. If $\eta \leqslant \gamma$ then $f^{s_\nu,\tau}_{\eta,\eta'} = f^{s,\tau}_{\eta,\eta'}$ and $A^{s_\nu,\tau}_\eta = A^{s,\tau}_\eta$, furthermore $a^{s_\nu,\tau}_{\eta,\eta'} = a^{s,\tau}_{\eta,\eta'}$ for all $\eta' \leqslant \gamma$.
3. If $\eta > \gamma$ then $\{\, w \in A^{s_\nu,\tau}_\eta \mid \bar\kappa^w \leqslant \bar\kappa^{w_\nu} \,\} \subset A^{s_{\nu+1},\tau}_\eta$ and $a^{s_{\nu+1},\tau}_{\eta,\eta'}|\bar\kappa^{w_\nu} = a^{s_\nu,\tau}_{\eta,\eta'}|\bar\kappa^{w_\nu}$ for all $\nu < \bar\kappa^s_\tau$.
4. $a^{\sigma(w_\nu)}_{\eta,\eta'} = a^{w_\nu}_{\eta,\eta'}$ for all η and η' such that $\zeta \geqslant \eta > \gamma \geqslant \eta'$.
5. $\mathrm{add}(s_{\nu+1}, \sigma(w_\nu)) \in D$.

Clause(2) implies that $U^{s_\nu,\tau}_\gamma = U^{s,\tau}_\gamma$ for all $\nu \leqslant \bar\kappa^s_\tau$, and with Clause (3) implies that the limits in Clause (1) exist.

To define $s_{\nu+1}$ and $\sigma(w_\nu)$, note that $\mathrm{add}(s_\nu, w_\nu) \leqslant^* \mathrm{add}(s, w_\nu)$, and therefore the hypothesis implies that there is $t \leqslant^* \mathrm{add}(s_\nu, w_\nu)$ such that $t \in D$. Fix such a condition t, and define

$$\begin{aligned}\sigma(w_\nu)\upharpoonright[\gamma_0,\gamma] &= t(\gamma)\\ \sigma(w_\nu)\upharpoonright(\gamma,\zeta] &= w_\nu\upharpoonright(\gamma,\zeta].\end{aligned}$$

Extend s_ν to $s_{\nu+1}$ by setting

$$\begin{aligned}f^{s_{\nu+1},\tau}_{\eta,\eta'} &= f^{s_\nu,\tau}_{\eta,\eta'} \cup f^{t,\tau}_{\eta,\eta'}\upharpoonright(\mathrm{domain}(f^{t,\tau}_{\eta,\eta'})\setminus \mathrm{domain}(f^{s_\nu,\tau}_{\eta,\eta'} \cup a^{s_\nu,\tau}_{\eta,\eta'}))\\ A^{s_{\nu+1},\tau}_\eta &= A^{t,\tau}_\eta \cup \{\, w \in A^{s_\nu,\tau}_\eta \mid \bar\kappa^w \leqslant \bar\kappa^{w_\nu} \,\}\end{aligned}$$

for all $\eta > \gamma$, leaving all other components of s_ν unchanged as required by clauses (2)–(4).

Then Clauses (3) and (2) hold automatically, since the elements required to be held constant do not exist in $\operatorname{add}(s_\nu, \sigma(w_\nu))$. Clause (5) holds since $\operatorname{add}(s_{\nu+1}, \sigma(w_\nu)) = \operatorname{add}(t, \sigma(w_\nu))$.

This completes the definition of $s_{\bar\kappa^s_\tau}$, and in order to extend it to s' let

$$\bar{s} = [\sigma]_{U^{s,\zeta}_\gamma} = i^{F_\gamma}(\sigma)(s(\tau)\uparrow\gamma). \tag{5}$$

and set

$$A^{s',\tau}_\eta = \begin{cases} A^{\bar{s},\tau}_\eta & \text{for } \eta < \gamma \text{ and} \\ \sigma[A^{s,\tau}_\gamma] & \text{for } \eta = \gamma, \text{ and} \end{cases}$$

$$f^{s',\tau}_{\eta,\eta'} = f^{\bar{s},\tau}_{\eta,\eta'}, \quad \text{and} \quad a^{s',\tau}_{\eta,\eta'} = a^{\bar{s},\tau}_{\eta,\eta'} \quad \text{for } \eta \leqslant \gamma.$$

Then $s' \leqslant^* s$, and if $w = \sigma(w') \in A^{s',\tau}_\gamma$ then $\operatorname{add}(s', w) \leqslant^* \operatorname{add}(s_\nu, \sigma(w')) \in D$. □

4.4 The Prikry property

Lemma 4.16. *1. Let φ be a sentence and s a condition in $P(\vec{F})$. Then there is an $s' \leqslant^* s$ such that s' decides φ.*

2. Let D be a dense subset of $P(\vec{F})$, and suppose $s \in P(\vec{F})$. Then there is an $s' \leqslant^* s$ and a finite $b \subseteq \zeta + 1$ such that any $s'' \leqslant s'$ with $b \subseteq \operatorname{domain}(s'')$ is a member of D.

Proof of Lemma 4.16. z In order to simplify notation, we assume that $\operatorname{domain}(s) = \{\zeta\}$. The full result then follows by an induction using Proposition 4.14. The proof is by induction on ζ: we assume as an induction hypothesis that the lemma is true of $\vec{F}\upharpoonright\zeta'$ for all $\zeta' < \zeta$.

The main part is the proof of the following claim:

Claim 4.16.1. *Suppose that $D \subseteq P(\vec{F})$ is dense and $s \in P(\vec{F})$ with* $\operatorname{domain}(s) = \{\zeta\}$*. Then there is $s' \leqslant^* s$ such that either $s' \in D$ or for some $\gamma < \zeta$*

$$\begin{aligned} s' \Vdash_{P(\vec{F})} (\exists w \in A^{s',\zeta}_\gamma) \big(&\operatorname{add}(s', w) \in \dot{G} \wedge \\ &\operatorname{add}(s', w)\upharpoonright(\gamma+1) \Vdash_{P(\vec{F}^w)} (\exists t \in \dot{G}^{P(\vec{F}^w)})\, (t \cup (\operatorname{add}(s', w)\upharpoonright\{\zeta\})) \in D\big). \end{aligned} \tag{6}$$

Proof of Claim 4.16.1. If there is $s' \in D$ such that $s' \leqslant^* s$ then we are done, so we can assume that there is no such s'. For each $\gamma < \zeta$, define

$$\begin{aligned} D^+_\gamma &= \{\, t \in P(\vec{F}) \mid t \Vdash (\exists t' \in \dot{G} \cap D)\ \operatorname{domain}(t') \subseteq (\gamma+1) \cup \{\zeta\} \,\} \\ D^-_\gamma &= \{\, t \in P(\vec{F}) \mid t \Vdash \neg(\exists t' \in \dot{G} \cap D)\ \operatorname{domain}(t') \subseteq (\gamma+1) \cup \{\zeta\} \,\} \\ E_\gamma &= \{\, t \in P(\vec{F}) \mid \forall t' \leqslant t((t' \in D \wedge \operatorname{domain}(t') \subseteq (\gamma+1) \cup \{\zeta\}) \\ &\qquad\qquad \implies t'\upharpoonright(\gamma+1) \cup t\upharpoonright\{\zeta\} \in D) \,\}. \end{aligned}$$

It will be enough to show that for all $\gamma < \zeta$ the set $(D^+_\gamma \cup D^-_\gamma) \cap E_\gamma$ is $\leqslant^*$-dense below any condition with domain $\{\gamma, \zeta\}$, for if we can do so then by Lemma 4.15 there is $s' \leqslant^* \sigma$ such that for each $\gamma < \zeta$ and $w \in A^{s',\zeta}_\gamma$, we have $\operatorname{add}(s, w) \in (D^+_\gamma \cup D^-_\gamma) \cap E_\gamma$. By shrinking the sets $A^{s',\zeta}_\gamma$ we can assume that for each γ, $\{\operatorname{add}(s', w) \mid w \in A^{s',\zeta}_\gamma\}$ is contained in one of $D^+_\gamma \cap E_\gamma$ or $D^-_\gamma \cap E_\gamma$. Since D is dense it follows that $\{\operatorname{add}(s', w) \mid w \in A^{s',\zeta}_\gamma\} \subseteq D^+_\gamma$ for some $\gamma < \zeta$, and it follows by Proposition 4.14(3) that s' satisfies the formula (6).

To see that $(D^+_\gamma \cup D^-_\gamma) \cap E_\gamma$ is $\leqslant^*$-dense below any $t \in P(\vec{F})$ with $\gamma \in \operatorname{domain}(t)$, first note that by Proposition 4.14(3), the set E_γ is $\leqslant^*$-dense below any condition t with $\gamma \in \operatorname{domain}(t)$. Now for any $t \in E_\gamma$, the induction hypothesis asserts that there is $t' \leqslant^* t \restriction (\gamma + 1)$ in $P(\vec{F}^{t,\gamma})$ such that

$$t' \parallel_{P(\vec{F}^{t,\gamma})} (\exists t'' \in \dot{G})\, t'' \cup t \restriction \{\gamma\} \in D.$$

Then $t' \cup t \restriction \{\zeta\}$ is in either D^+_γ or in D^-_γ. $\square$

We now apply Claim 4.16.1 to complete the proof of Lemma 4.16. For the first clause, let D be the set of conditions t such that $t \parallel \varphi$. If there is $s' \leqslant^* s$ in D then we are done, so by Claim 4.16.1 we can assume that there is $s' \leqslant^* s$ and $\gamma < \zeta$ such that for all $w \in A^{s',\zeta}_\gamma$

$$\operatorname{add}(s', w) \Vdash \big(\exists t \in \dot{G} \cap D\; (\operatorname{domain}(t) \subseteq (\gamma + 1 \cup \{\zeta\}) \wedge t(\zeta) = \operatorname{add}(s', w)(\zeta)\big).$$

By the induction hypothesis, it follows that there is $t_w \leqslant^* \operatorname{add}(s', w) \restriction (\gamma + 1)$ in $P(\vec{F}^w)$ such that $t_w \cup \operatorname{add}(s', w)] \restriction \{\zeta\} \parallel \varphi$. By shrinking $A^{s',\zeta}_\gamma$ if necessary, we can assume that φ is decided the same way by $t_w \cup \operatorname{add}(s', w) \restriction \{\zeta\}$ for every $w \in A^{w,s'}_\gamma$. Then $s' \leqslant^* s$ decides φ.

The second clause of Lemma 4.16 is proved similarly, using the given dense set D. $\square$

Corollary 4.17. *Suppose that $\dot{x}$ is a $P(\vec{F})$-name for a subset of λ, $\gamma \leqslant \zeta$, and s is a condition with $\gamma \in \operatorname{domain}(s)$ such that $\lambda < \bar{\kappa}^s_\gamma$. Then there is $s' \leqslant^* s$ such that $s' \Vdash \dot{x} \in M[\{t \restriction \gamma \mid t \in \dot{G}\}]$.*

Proof. By Proposition 4.14 we can factor $P(\vec{F})\|_s$ as $P(\vec{F} \restriction \gamma)\|_{s \restriction \gamma} \times R$. By the remark following Proposition 4.14, Theorem 4.16 holds of R, and since $(R, \leqslant^*)$ is $|\lambda \times P(\vec{F} \restriction \gamma)\|_{s \restriction \gamma}|$-closed the conclusion follows. $\square$

Corollary 4.18. *Forcing with $P(\vec{F})$ does not collapse any cardinal λ which is not in the set $\bigcup_{\gamma \leqslant \zeta} [\bar{\kappa}^{++}_\gamma, \bar{\kappa}^{+\omega_1}_\gamma)$.*

Proof. Suppose that s is a condition which forces that $|\lambda| = \lambda' < \bar{\kappa}_\gamma$. Then by Corollary 4.17 it follows that $s \restriction (\gamma + 1) \Vdash_{P(\vec{F}^{s,\gamma})} |\lambda| = \lambda'$. Thus we can assume without loss of generality that $\lambda > \kappa = \bar{\kappa}_\zeta$. Furthermore, since $|P(\vec{F})| < \kappa^{+\omega_1}$ we can assume that $\lambda < \kappa^{+\omega_1}$.

Now $\lambda \in (\kappa, \kappa^{+\omega_1})$, so it only remains to show that κ^+ is not collapsed. To see this, let $\dot{f}$ be the name of a function $f \colon \kappa \to \kappa^+$.

For each $\gamma < \zeta$, let D_γ be the set of conditions t with $\gamma \in \mathrm{domain}(t)$ such whenever $\xi < \bar{\kappa}^t_\gamma$, $t' \leqslant t$, and $\mathrm{domain}(t') \subseteq \mathrm{domain}(t) \cup (\gamma+1)$, if there is α such that $t' \Vdash \dot{f}(\xi) = \alpha$, then $t' \restriction (\gamma+1) \cup t \restriction (\gamma, \delta] \Vdash \dot{f}(\xi) = \alpha$. By Lemma 4.14 the sets D_γ are open and $\leqslant^*$-dense below any condition t with $\gamma \in \mathrm{domain}(t)$. By Lemma 4.15, then, there is $s'' \leqslant^* s'$ such that $\mathrm{add}(s'', w) \in D_\gamma$ for any $\gamma < \zeta$. Then $\mathrm{range}(f)$ is contained in $\{\, \alpha \mid \exists t \leqslant s'' \mid t \restriction [\gamma, \zeta] = s'' \restriction [\gamma, \zeta] \,\}$, which is a set in the ground model of size κ. □

In the forcing of Gitik from which this forcing is derived, a preliminary forcing is used to define a morass-like structure which guides the main forcing so as to prevent any collapse. The preliminary forcing is omitted here as unnecessary to the present purpose; however as a consequence we do not know whether the cardinals of M_Ω in $(\kappa^+, \kappa^{+\omega_1})$ which are excepted in Lemma 4.18 remain cardinals in the Chang model. However this is not a significant question with the present choice of the model M: the cardinals in question can be expected actual indiscernibles for the Chang model.

4.5 Introducing the equivalence relation

We now proceed to the second part of the definition of the forcing by introducing a variant of Gitik's equivalence relation $\leftrightarrow$ on $P(\vec{F})$, which is based on the following equivalence relation on $[\mathrm{supp}(F_\gamma)]^{\bar{\kappa}_\gamma}$:

Definition 4.19. Suppose that $\vec{F}$ is a suitable sequence of extenders of length at least $\gamma+1$ on a cardinal λ, and $b, b' \subset [\mathrm{supp}(F_\gamma)]^\lambda$. Then $b \leftrightarrow_0 b'$ if $\mathrm{otp}(b) = \mathrm{otp}(b')$ and, setting $Y = \bigcup_{\nu<\gamma} \mathrm{supp}(F_\nu)$, we have $b \cap Y = b' \cap Y$ and $F_\gamma \wp (Y \cup b) = F_\gamma \wp (Y \cup b')$.

If $n \geqslant 0$ then we say $b \leftrightarrow_{n+1} b'$ if for all $c \supseteq b$ in $[\mathrm{supp}(F_\gamma)]^\lambda$ there is $c' \supseteq b'$ in $[\mathrm{supp}(F_\gamma)]^\lambda$ such that $c \leftrightarrow_n c'$, and for all $c' \supseteq b'$ there is $c \supseteq b$ such that $c \leftrightarrow_n c'$.

Definition 4.20. We write $\mathcal{N}$ for the set of sequences $\vec{n} \in {}^\zeta\omega$ such that $\{\, \iota < \zeta \mid n_\iota < m \,\}$ is finite for each $m \in \omega$.

If $\vec{a}$ and $\vec{a}'$ are arrays of Prikry functions as in Tableau 1 and $\vec{n} \in \mathcal{N}$, then we say $\vec{a} \leftrightarrow_{\vec{n}} \vec{a}'$ if the patterns of $\vec{a}$ and $\vec{a}'$ are the same, and $\mathrm{range}(a_{\gamma+1,\gamma}) \leftrightarrow_{n_\gamma} \mathrm{range}(a'_{\gamma+1,\gamma})$ for all γ such that these functions appear in the tableau.

We say that $\vec{a} \leftrightarrow \vec{a}'$ if $\vec{a} \leftrightarrow_{\vec{n}} \vec{a}'$ for some $\vec{n} \in \mathcal{N}$.

Note that $a_{\gamma+1,\gamma}$, together with the pattern of $\vec{a}$, determines the rest of the column $\langle\, a_{\nu,\gamma} \mid \tau \geqslant \nu > \gamma \,\rangle$.

Definition 4.21. The extension of the relation $\leftrightarrow_{\vec{n}}$ to members of P^*_γ and $P^*_{\eta,\gamma}$ is by recursion on γ. Assume that $\leftrightarrow_{\vec{n}}$ has already been defined on $P^*_{\eta,\gamma'}$ and $P^*_{\gamma'}$ for all $\gamma' < \gamma$. For $w, w' \in P^*_{\eta,\gamma}$ we say

$$w \leftrightarrow_{\vec{n}} w' \iff \left(w \restriction [\gamma_0, \eta] \leftrightarrow_{\vec{n}} w' \restriction [\gamma_0, \eta] \;\;\&\;\; (\forall \mu > \eta \geqslant \mu')\; a^w_{\mu,\mu'} = a^{w'}_{\mu,\mu'} \right),$$

where $w\restriction[\gamma_0,\eta]$ and $w'\restriction[\gamma_0,\eta]$ are being compared as members of P^*_η.

For $t, t' \in P^*_\gamma$, we say $t \leftrightarrow_{\vec{n}} t'$ if and only if the following four conditions are satisfied:

1. $\bar{\kappa}^t = \bar{\kappa}^{t'}$, $\vec{F}^t = \vec{F}^{t'}$.
2. $f^t_{\nu,\nu'} = f^{t'}_{\nu,\nu'}$ for all ν, ν' for which they are defined.
3. $\vec{a}^t \leftrightarrow_{\vec{n}} \vec{a}^{t'}$.
4. $\{ [w]_{\leftrightarrow_{\vec{n}}} \mid w \in A^t_\eta \} = \{ [w]_{\leftrightarrow_{\vec{n}}} \mid w \in A^{t'}_\eta \}$ for all $\eta < \gamma$.

Finally, if $s, s' \in P(\vec{F})$ then $s \leftrightarrow_{\vec{n}} s'$ if domain(s) = domain(s') and $s(\gamma) \leftrightarrow_{\vec{n}} s'(\gamma)$ for all γ in the common domain.

In all cases we say that $\leftrightarrow$ holds if there is some $\vec{n} \in \mathcal{N}$ such that $\leftrightarrow_{\vec{n}}$ holds.

It is easy to see that $\leftrightarrow$ is an equivalence relation. As was pointed out earlier, its purpose is to disassociate the Prikry indiscernible $h_{\nu,\nu'}(\xi) = f_{a,a'}(\xi)$ from any particular choice of the ordinal $a(\xi)$ for which it is an indiscernible.

Proposition 4.22. *Suppose that $s \leftrightarrow s'$, $w \in A^{s,\tau}_\gamma$, $w' \in A^{s',\tau}_\gamma$ and $w \leftrightarrow w'$. Then $f^{\mathrm{add}(s,w),\tau}_{\eta,\eta'} = f^{\mathrm{add}(s',w'),\tau}_{\eta,\eta'}$ for all $\tau \in \mathrm{domain}(s)$ and $\gamma_0 \leqslant \eta' < \eta \leqslant \tau$.*

Proof. This is immediate from the definition except in the case that $\eta > \gamma \geqslant \eta'$. If $\eta' = \gamma$ then it follows from the requirement if Definition 4.21that $a^w_{\eta,\eta'} = a^{w'}_{\eta,\eta'}$. For $\eta' < \gamma$ also relies on the fact that $f_{a,a'}(\xi) = h_{\gamma,\eta'}(\xi')$, using the second form of equation (3), and hence depends only on the domain of $a^{w'}_{\gamma,\eta'}$, not (explicitly) on its value. $\square$

Proposition 4.23. *Suppose that $\mathrm{add}(s,\vec{z}) \leqslant s \leftrightarrow_{\vec{n}} t$. Then there is $\vec{w}$ such that $\mathrm{add}(s,\vec{z}) \leftrightarrow_{\vec{n}} \mathrm{add}(t,\vec{w}) \leqslant t$.*

Proof. We show that this is true when $\vec{z}$ has length one. An induction will then show that it is true in general. Thus suppose that $z \in A^{s,\tau}_\gamma$ and $\mathrm{add}(s,z) \leqslant s \leftrightarrow_{\vec{n}} t$. By Definition 4.21(4) there is $w \in A^{t,\tau}_\gamma$ such that $z \leftrightarrow_{\vec{n}} w$. Then $\mathrm{add}(t,w)(\gamma) = w\restriction[\gamma_0,\gamma] \leftrightarrow_{\vec{n}} z\restriction[\gamma_0,\gamma] = \mathrm{add}(s,z)(\gamma)$; and Proposition 4.22 implies that $\mathrm{add}(t,w)(\tau) \leftrightarrow_{\vec{n}} \mathrm{add}(s,z)(\tau)$.

Since these are the only values of s and t which are changed, it follows that $\mathrm{add}(s,z) \leftrightarrow_{\vec{n}} \mathrm{add}(t,w)$. $\square$

Proposition 4.24. *Suppose $s' \leqslant^* s \leftrightarrow_{\vec{n}} t$, and that $n_\nu > 0$ for all $\nu \notin \mathrm{domain}(s)$. Then there is $t' \leqslant^* t$ such that $s' \leftrightarrow_{\vec{m}} t'$, where $m_\nu = n_\nu - 1$ if $n_\nu > 0$, and $m_\nu = 0$ otherwise,*

Proof. We will prove that the lemma is true for s', s and t in P^*_γ with the assumption that $n_\nu > 0$ for all ν in the interval $\gamma_0 \leqslant \nu < \gamma$; this will imply that it is true for s', s and t in $P(\vec{F})$. The proof is by induction on γ.

By the definition of $\leqslant^*$ and $\leftrightarrow$ all of s', s, t' and t must agree on their values of $\bar{\kappa}$ and $\vec{F}$, and $f^{t'}$ must be equal to $f^{s'}$. This leaves the functions $a^{t'}_{\nu',\nu}$ and sets $A_\nu^{t'}$ to be defined. To define $\vec{a}^{t'}$, pick for each ν in the interval $\gamma_0 \leqslant \nu < \gamma$ some $b \supseteq a^{t,\gamma}_{\nu+1,\nu}$ such that $a^{s'}_{\nu+1,\nu} \leftrightarrow_{m_\nu} b$. This is possible by the definition of $\leftrightarrow_{m_\nu+1}$. Now set $a^{t'}_{\nu+1,\nu} = b$ and define $a^{t'}_{\nu',\nu}$ for $\gamma \geqslant \nu' > \nu + 1$ by applying the pattern of $\langle a^{s'}_{\nu',\nu} \mid \gamma \geqslant \nu' > \nu \rangle$.

Finally, set

$$A_\nu^{t'} = \{ w' \mid \exists w \in A_\nu^t \ w' \leqslant^* w \ \ \& \ \ \exists v' \in A_\nu^{s'} \ w' \leftrightarrow_{\vec{m}} v' \}.$$

To see that $\bar{A}_\nu^{t'} = \bar{A}_\nu^{s'}$, observe that $s' \leqslant^* s$ implies that for all $v' \in A_\nu^{s'}$ there is $v \in A_\nu^s$ such that $v' \leqslant^* v$. Then $s \leftrightarrow_{\vec{n}} t$ implies that there is w in A_ν^t such that $w \leftrightarrow_{\vec{n}} w'$, and the induction hypothesis implies that there is $w' \leqslant^* w$ with $w' \leftrightarrow_{\vec{m}} v'$. □

Definition 4.25. We write $[s]$ for $[s]_\leftrightarrow = \{ t \mid s \leftrightarrow t \}$. The ordering on $P(\vec{F})/\leftrightarrow$ is the least transitive relation such that $[s] \leqslant [t]$ whenever $s \leqslant t$ or $s \leftrightarrow t$.

Proposition 4.26. *Suppose $[t] = [s]$ and $t' \leqslant t$. Then there are $s'' \leqslant s$ and $t'' \leqslant t'$ such that $[s''] = [t'']$.*

Proof. Suppose that $t \leftrightarrow_{\vec{n}} s$. By using a further extension $t'' = \operatorname{add}(t', \vec{w})$ we can arrange that $\{ \nu \mid n_\nu = 0 \} \subseteq \operatorname{domain}(t'')$. By Proposition 4.12 there is $\vec{z}$ so that $t'' \leqslant^* \operatorname{add}(t, \vec{z}) \leqslant t$. By Proposition 4.23 it follows that there is $\vec{w}$ so that $\operatorname{add}(t, \vec{z}) \leftrightarrow_{\vec{n}} \operatorname{add}(s, \vec{w}) \leqslant s$. Finally it follows by Proposition 4.24 that there is $s'' \leqslant^* \operatorname{add}(s, \vec{w})$ so that $s'' \leftrightarrow t''$. □

Proposition 4.27. *Suppose that $[t] \leqslant [s]$. Then there is a condition $q \leqslant s$ such that $[q] \leqslant [t]$.*

Proof. If $[t] \leqslant [s]$ then there is a sequence $\langle t_1, \ldots, t_{2n} \rangle$ as in the first row of the following equation:

$$\begin{tikzcd}
& t \leqslant t_1 \arrow[r, leftrightarrow] & t_2 \leqslant t_3 \leftrightarrow \cdots \leftrightarrow t_{2n} = s \\
q \arrow[ur, phantom, "\leqslant\ \leftrightarrow"] \arrow[rr, leftrightarrow] & & q' \arrow[u, phantom, "\leqslant\ \leftrightarrow"] \arrow[u, leftrightarrow, bend right]
\end{tikzcd} \tag{7}$$

We prove the proposition by induction on n. If $n = 0$ then $t = s$, so we can assume that $n > 0$. Then the induction hypothesis asserts that there is q' as shown in diagram (7), and then Proposition 4.26 implies that there is q as in diagram (7). □

Corollary 4.28. *$P(\vec{F})$ is forcing equivalent to $(P(\vec{F})/\leftrightarrow) * \dot{R}$ where $\dot{R}$ is a $P(\vec{F})/\leftrightarrow$-name for a partial order.* □

Corollary 4.29. *Forcing with $P(\vec{F})/\leftrightarrow$ does not collapse any cardinal which is not in the set $\bigcup_{\gamma \leqslant \zeta} [\bar{\kappa}_\gamma^{++}, \bar{\kappa}_\gamma^{+\omega_1})$.*

Proof. By Corollary 4.18 this is true in the extension by $P(\vec{F}) = (P(\vec{F})/\leftrightarrow) * \dot{R}$; hence it is certainly true in the extension by $P(\vec{F})/\leftrightarrow$. □

4.6 Constructing a generic set

Much of the argument in this subsection is basically the same as Carmi Merimovich's first genericity construction in [Mer07, Theorem 5.1]. In order to construct a M_B-generic set we need to move outside of M_B: we work in $V[h]$, where h is a generic collapse of $\mathcal{R}$ onto ω_1 so that $|M[h]| = \omega_1^{M[h]} = \omega_1$. Since this Levy collapse does not add countable sequences of ordinals, the Chang model is unchanged, the ordering $\leqslant^*$ of $P(\vec{N}\restriction\zeta)$ is still countably complete, and M is still closed under countable sequences.

Before continuing, it may be useful to look briefly at the relation between $\mathbb{C}^{M_B[G]}$ and the actual Chang model. Since M_B is not transitive, the fact that $M_B|\Omega \subseteq V$ does not immediately imply that $\mathbb{C}^{M_B[G]}$ is isomorphic to a submodel of $\mathbb{C}$. The ultimate conclusion will be that this is true, but the proof will be in the following Section 4.7 as part of the proof of the Main Lemma 3.11. The only case in which this isomorphism is immediately clear, given the result in this section asserting that $M_B[G]|\Omega$ contains all of its countable subsets, is the case $B = B(\delta) = \langle \kappa_\alpha \mid \alpha < \delta \rangle$: in this case $M_B[G]|\Omega$ is transitive and it follows that $\mathbb{C}^{M_B[G]} = \mathbb{C}_{\kappa_\delta}$. It is also easy to see that if B and B' have the same order type, then $\mathbb{C}^{M_B[G]} \cong \mathbb{C}^{M_{B'}[G']}$. If B is suitable, then the definition of $\mathbb{C}_B$ essentially says that it is equal to $\mathbb{C}^{M_B[G]}$; however it is not immediately clear what relationship $\mathbb{C}^{M_B[G]}$ may have to the Chang model itself. In the case that B is limit suitable, it is not even immediately clear that $\mathbb{C}_B$ is definable in $M_B[G]$.

The main result of the current section is the following lemma:

Lemma 4.30. *Let h be a generic collapse of $\mathcal{R}$ onto ω_1 with countable conditions, and let B be a countable subset of I with* $\operatorname{otp}(B) = \zeta$*. Then there is, in $V[h]$, an $i_\Omega(M_B)$-generic set $G \subseteq i_\Omega(P(\vec{N}\restriction\zeta)/\leftrightarrow)$ such that every countable subset of M_B is contained in $M_B[G]$. Therefore, if B is suitable then $\mathbb{C}^{M_B[G]} = \mathbb{C}_B$.*

Proof. We define a partial order R. Our assumptions on M are sufficiently generous that the definition of R can be made inside M, using $\langle N_\xi \cap H_\tau^M \mid \xi < \omega_1 \rangle$, for some sufficiently large cardinal τ of M, instead of $\langle N_\xi \mid \xi < \omega_1 \rangle$.

Definition 4.31. $R = \bigcup_{\xi<\omega_1} R_\xi$, where R_ξ is defined as follows: The members of R_ξ are the pairs $([s], b)$ such that $[s] \in P(\vec{E}\restriction\delta)/\leftrightarrow$ is a condition with $\operatorname{domain}(s) = \{\zeta\}$ and $b = \langle b_\gamma : \gamma < \zeta \rangle$ where each b_γ is a function in N_ξ satisfying the following three conditions:

1. $\operatorname{domain}(b_\gamma) = \operatorname{domain}(a^{s,\zeta}_{\gamma+1,\gamma})$ for each $\gamma < \zeta$,
2. $\operatorname{range}(b_\gamma) \subset [\kappa, \kappa^{+\omega_1})$ for each $\gamma < \zeta$, and
3. $\langle a^{s,\zeta}_{\gamma+1,\gamma} \mid \gamma < \zeta \rangle \leftrightarrow b$.

The ordering of R is $(s', b') \leqslant (s, b)$ if $[s'] \leqslant [s]$ in $P(\vec{E}\restriction\zeta)/\leftrightarrow$ and $b'_\gamma \supseteq b_\gamma$ for all $\gamma < \zeta$.

Clause(3) requires some further discussion, since $b \notin N_\gamma$. The following definition gives a representation of the $\leftrightarrow_n$-equivalence class of a which makes sense in this context.

Definition 4.32. Fix $\gamma < \omega_1$ and write $S = \bigcup_{\gamma' < \gamma} \operatorname{supp}(E_{\eta'})$. The the functions $t_{\gamma,n}$ for $n \in \omega$ are defined as follows:

1. If $a \in [\operatorname{supp}(E)]^\kappa$ then $t_{\gamma,0}(a) = \{ (\beta, x) \in S \times \mathcal{P}(V_\kappa) \mid x \in E_{(\beta,a)} \}$.
2. $t_{\gamma,n+1}(a)$ is the set of triples (ξ, y, X) such that $y \subseteq \xi < \kappa^+$ and for some $a' = \{ \alpha_\iota \mid \iota < \xi \} \subseteq \operatorname{supp}(E)$ such that $a = \{ \alpha_\iota \mid \iota \in y \}$ we have $t_{\gamma,n}(a') = X$.

Proposition 4.33. *1. If $a \in [\operatorname{supp}(E_\gamma)^\kappa$ and $\gamma \leqslant \xi < \omega_1$ then $(t_{\gamma,n}(a))^{N_\gamma} = (t_{\gamma,n}(a))^{N_\xi} = (t_{\gamma,n}(a))^M$.*

2. If $a, a' \in [\operatorname{supp}(E_\gamma)]^\kappa$ then $a \leftrightarrow_n a'$ if and only if $t_{\gamma,n}(a) = t_{\gamma,}(a')$.

3. If $a \in [\operatorname{supp}(E_\gamma)]^\kappa$, $t_{\gamma,n+1}(a) = t_{\gamma,n+1}(b)$ and $b \subseteq b' \in [\operatorname{supp}(E)]^\kappa$ then there is a' such that $a \subseteq a' \in [\operatorname{supp}(E_\gamma)]^\kappa$ and $t_{\gamma,n}(a') = t_{\gamma,n}(b')$. □

Lemma 4.34. *1. $\{ ([s], b) \mid s \in D \}$ is dense in R for each $\leqslant^*$-dense set $D \subseteq P(\vec{E}\restriction\zeta)$ in M.*

2. Suppose $\nu < \zeta$ and $\eta \in [\kappa, \kappa^{+\omega_1})$, and define $b_{\zeta,\nu}$ by applying the pattern of $a^{s,\zeta}_{\zeta,\nu}$ to b_ν. Then $\{ ([s], b) \mid (\eta \in \operatorname{range}(b_{\zeta,\nu})) \}$ is dense in R.

Proof. To see that $\{ ([s], b) \mid s \in D \}$ is dense in R whenever $D \subseteq P(\vec{N}\restriction\zeta)$ is $\leqslant^*$-dense, let $([s], b) \in R$ be arbitrary. Let $\vec{a} = \langle a^{s,\zeta}_{\gamma+1,\gamma} \mid \gamma < \zeta \rangle$. We may assume that $a_\gamma \leftrightarrow_1 b_\gamma$ for each $\gamma < \zeta$; if not, then replace a_γ with some a'_γ such that $a'_\gamma \leftrightarrow_0 a_\gamma$ and $a'_\gamma \leftrightarrow_1 b_\gamma$. This is possible by the elementarity of the structures N_ξ, since b_γ has the desired properties. This change only involves finitely many of the functions a_γ, so the condition obtained from s by making this substitution is still in $[s]$.

Now pick $s' \leqslant^* s$ in D. Because of the assumption we made on $\vec{a}$, there is $b' \leftrightarrow a^{s',\zeta}$ such that $([s'], b') \leqslant ([s], b)$.

The proof of clause 2 is similar. Fix $([s], b) \in R$, making the same assumption on s as before. Now fix ξ so that $\{b, \eta\} \in N_\xi$ and and extend b to $b' \in N_\xi$ by setting $b'_\nu(\alpha) = \eta$ where α is chosen large enough so that it is not in the domain of any function in s. Then there is $\vec{a}' \supset \vec{a}$ so that $\vec{a}' \leftrightarrow b'$. Now extend s to s', choosing the pattern of $\langle a^{s',\zeta}_{\gamma,\nu} \mid \zeta \geqslant \gamma > \nu \rangle$ by including α in $\operatorname{domain}(a^{s',\zeta}_{\gamma,\nu})$ for all γ. Then $\alpha \in \operatorname{domain}(b_{\zeta,\nu})$ and hence $\eta \in \operatorname{range}(b_{\zeta,\nu})$. □

The ordering $(P(\vec{N})/\leftrightarrow, \leqslant^*)$ is not countably complete: it is easy to find an infinite descending sequence of conditions $\langle [s_n] \mid n < \omega \rangle$ such that any lower bound would require an ultrafilter concentrating on non-well founded sets of ordinals. However the partial order R is countably complete due to the guidance of the second coordinate b:

Lemma 4.35. *The partial order R is countably closed.*

Proof. Suppose that $\langle ([s_n], b_n) \mid n < \omega \rangle$ is a descending sequence in R. We define a lower bound $([s_\omega], b_\omega)$ for this sequence. The definition of R determines $b_{\omega,\nu} = \bigcup_{n<\omega} b_{n,\nu}$, and determines all of s_ω except for the functions $a^{s_\omega,\zeta}_{\nu+1,\nu}$ and sets $A^{s_\omega,\zeta}_{\nu}$.

Let us write $a_{n,\nu}$ for $a^{s_n,\zeta}_{\nu+1,\nu}$. The domain and pattern of $\vec{a}_{\omega,\nu}$ is immediately determined: $\operatorname{domain}(a_{\omega,\nu}) = \operatorname{domain}(b_{\omega,\nu}) = \bigcup_{n<\omega} \operatorname{domain}(a_{n,\nu})$, and the pattern of $\vec{a}_\omega$ is determined by the requirement that the pattern of $a_{\omega,\nu} \restriction \operatorname{domain}(a_{n,\nu}$ is the same as the pattern of $a_{n,\nu}$. Pick any $\vec{n} = \langle n_\nu \mid \nu < \zeta \rangle \in \mathcal{N}$, and for each $\nu < \zeta$ pick $a_{\omega,\nu} \in N_\nu$ so that

$$a_{\omega,\nu} \restriction \operatorname{domain}(a_{n,\nu}) \leftrightarrow_{k_{n,\nu}} a_{n,\nu} \quad \text{and} \quad a_{\omega,\nu} \leftrightarrow_{n_\nu} b_{\omega,\nu}$$

where $k_{n,\nu} \in \mathcal{N}$ is chosen so that $a_{n,\nu} \leftrightarrow_{k_{n,\nu}} b_{n,\nu}$. This is possible by Proposition 4.33 since $b_{\omega,\nu}$ satisfies these conditions.

Now define the sets $A^{s_\omega,\zeta}_{\nu}$ as in the proof of Proposition 4.24. Then $([s_\omega], b_\omega) \in R$ and $([s_\omega], b_\omega) \leqslant ([s_n], b_n)$ for each $n \in \omega$. □

We are now ready to construct the M_B-generic subset $G \subseteq i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$, where $\zeta = \operatorname{otp}(B)$. To slightly simplify the notation, we will assume that $B = B(\zeta) = \{ \kappa_\gamma \mid \gamma < \zeta \}$, so that the constructed generic set will have $\bar{\kappa}_\gamma = \kappa_\gamma$. For an arbitrary set B there is an isomorphism $\pi_{B(\zeta),B} \colon M_{B(\zeta)} \cong M_B$, so that if $G_{B(\zeta)}$ is the constructed $M_{B(\zeta)}$-generic set then $G_B = \pi_{B(\zeta),B}[G_{B(\zeta)}]$ will be the required M_B-generic set.

Definition 4.36. To construct the set $G \subseteq i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$, first construct a M-generic set $H \subset R$ in $V[h]$. This is possible by Lemma 4.35 since $|M|^{V[h]} = \omega_1$ and and ${}^\omega M \subseteq M$. Now for each $([s], b) \in H$ and each finite increasing sequence $\langle \gamma_i \mid i < n \rangle$ of ordinals $\gamma_i < \zeta$, define a sequence $\vec{w}(s, \vec{\gamma})$ by setting

$$\vec{w}(s,\vec{\gamma}) = \langle i_{\gamma_i}(\bar{w}_i) \mid i < n \rangle, \quad \text{where}$$
$$\bar{w}_i = \Big(\kappa, \vec{E}\restriction[\gamma_{i-1},\gamma_i), \operatorname{repl}(z^{s,\zeta}\restriction[\gamma_{i-1},\gamma_i), a_{\gamma_i+1,\gamma_i}, b_{\gamma_i}), \vec{A}\restriction[\gamma_{i-1},\gamma_i)\Big). \tag{8}$$

Here we take $\gamma_{-1} = -1$, and we write $\operatorname{repl}(z, a_{\gamma+1,\gamma}, b_\gamma)$ for the tableau which is identical to z except that $a^z_{\gamma+1,\gamma}$ is replaced by b_γ, and hence each entry $a^z_{\nu,\gamma}$ in the column above $a^z_{\gamma+1,\gamma}$ is replaced by $b_\gamma \restriction \operatorname{domain}(a^z_{\nu,\gamma})$. Finally, set

$$G = \{ s' \mid (\exists([s],b) \in H)\, \exists\vec{\gamma}\ \operatorname{add}(i_\Omega(s), \vec{w}(s,\vec{\gamma})) \leqslant s' \} \tag{9}$$

Note that the effect of the replacement used in equation (8) is that a condition $\operatorname{add}(i_\Omega(s), \vec{w}) \in G$ forces that $h_{\zeta,\gamma_i}(i_\Omega(\xi)) = i_{\gamma_i}(b_\nu(\xi))$ for each $\xi \in \operatorname{domain}(a^{s,\zeta}_{\zeta,\nu})$. By Proposition 4.34, it follows that every generator of M_B will be given a name of this form.

Our verification that G behaves as expected relies on a system of standard names for members of M_B and $\mathbb{C}^{M_B[G]}$. Note that we use standard *forcing* names even for members of M_B.

Definition 4.37. A *standard forcing name* for a generator β of M_B is a term $h_{\zeta,\gamma}(i_\Omega(\xi))$ where $\gamma \in B$, $\xi \in [\kappa, \kappa^+)^M$ and $h^G_{\zeta,\gamma}(\xi) = \beta$. This name is *established* by a condition $([s], b) \in R$ if $\beta = i_\gamma(b(\xi))$ and $\xi \in \operatorname{domain}(a^{s,\zeta}_{\zeta,\gamma})$.

We extend this terminology to members of M_B and members of $\mathbb{C}$: a standard forcing name for $x \in M_B$ is a term of the form $i_\Omega(f)(\vec{\beta})$ where $\vec{\beta}$ is represented by a finite sequence of standard forcing names, and a standard name for a set in $\mathbb{C}$ is one of the form $\{\, z \in \mathbb{C}\iota \mid \mathbb{C}_\iota \models \varphi(z, \vec{\tau}) \,\}$ where ι is represented by a standard forcing name and $\vec{\tau}$ is represented by a countable sequence of such names.

Claim 4.37.1. *G is an M_B-generic subset of $P(i_\Omega(\vec{E}\restriction\zeta))$.*

Proof. The requirement that $\langle\, a^{s,\zeta}_{\zeta,\gamma} \mid \gamma < \zeta \,\rangle \leftrightarrow b$ implies that $w_i(\vec{s}, \vec{\gamma}) \in i_\Omega(A^{s,\zeta}_\gamma)$ and therefore $\operatorname{add}(s, \vec{w}(s, \vec{\gamma})) \leqslant i_\Omega(s)$, and it is straightforward to verify that the members of G are compatible.

To verify that G is generic, let D be an arbitrary M_B-generic subset of $i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$. Then $D = i(d)(\vec{\beta})$ for a function $d \in M$ and sequence $\vec{\beta} = \langle\, \beta_i \mid i < n \,\rangle$ of generators, say $\beta_i = i_{\gamma_i}(\bar{\beta}_i)$. Thus there is a standard forcing name $i(d)(\langle\, h_{\zeta,\gamma_i}(i_{\gamma_i}(\xi_i)) \,\rangle)$ for D. Let $([s], b) \in H$ be a condition establishing this name.

Since D is dense in $i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$,

$$A = \{\, \vec{\nu} \mid d(\vec{\nu}) \quad \text{is a dense subset of} \quad P(\vec{E}\restriction\zeta/\leftrightarrow) \,\} \in \prod_{i<n} E^{\gamma_i}_{\bar{\beta}_i},$$

so we may assume that $A \subseteq \prod_{i<n} A^{s,\zeta}_{\gamma_i}$.

Let $D' \subseteq P(\vec{E}\restriction\zeta/\leftrightarrow)$ be the set of conditions $[t]$ such that $[t] \Vdash [t] \in d(\langle\, h_{\zeta,\gamma_i}(\xi_i) \mid i < n \,\rangle)$. This is dense below $[s]$, since $[s] \Vdash d(\langle\, h_{\zeta,\gamma_i}(\xi_i) \mid i < n \,\rangle)$ is dense. Then by Lemma 4.16(2), there is an $s' \leqslant^* s$ and a finite $c \subseteq \zeta$ such that any $t \leqslant s'$ with $c \subseteq \operatorname{domain}(t)$ is in D', and it follows that $\operatorname{add}(i(s'), \vec{w}(s', c \cup \vec{\gamma})) \in D \cap G$. $\square$

Claim 4.37.2. *The model $M_B[G]|\Omega$ contains all countable sequences of its ordinals.*

Proof. It is sufficient to show that every countable sequence of generators of M_B is in $M_B[G]$. Thus let $\vec{\beta} = \langle\, i_{\gamma_i}(\bar{\beta}_i) \mid i \in \omega \,\rangle$. By Proposition 4.34(2), there is a condition $([s], b) \in H$ and a sequence $\vec{\xi}$ with $b_{\gamma_i}(\xi_i) = \bar{\beta}_i$ for each $i \in \omega$. Then $\vec{\beta} = \langle\, h_{\zeta,\gamma_i}(\xi_i) \mid i \in \omega \,\rangle \in M_B[G]$. $\square$

This completes the proof of Lemma 4.30. $\square$

Before turning to the next section, which generalizes the construction of this section in order to prove Main Lemma 3.11, we make two observations. The first asserts that, although $P(\vec{F})$ differs from Prikry forcing in that conditions in the forcing have incompatible $\leqslant^*$-extensions, such extensions cannot force incompatible information about the Chang model.

Corollary 4.38. *Suppose φ is a formula and that all parameters of φ are given by standard forcing names which are established by the condition $([s], b) \in R$. Then the following are equivalent:* (i) $\mathbb{C}_{B(\zeta)} \models \varphi$, (ii) *There is $s' \leqslant^* s$ such that $i_\Omega([s']) \Vdash_{i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)} \varphi$, and* (iii) $([s], b) \Vdash_R \mathbb{C}^{M_{B(\zeta)}[\dot{G}]} \models \varphi$.

Proof. When G is constructed as in the proof of Lemma 4.30, Clause (i) holds if and only if $\mathbb{C}^{M_B[G]} \models \varphi$. The construction can begin by choosing any condition as a member of H, so this is equivalent to Clause (iii) since if Clause (iii) is false then there is $([s], b') \leqslant ([s], b)$ which forces $\mathbb{C}^{M_B}[G] \models \neg\varphi$. Finally, Clause (ii) is equivalent since by Lemma 4.16 there is $s' \leqslant^* s$ which decides the question, and as with Clause (iii) it can only be decided one way. $\square$

Clause (i) uses the sequence $B(\zeta)$ instead of allowing an arbitrary B because the other two clauses are talking about facts which are internal to M_B (or, perhaps better, internal to $M_B[G]$) in the sense that they do not take any account of gaps in B.

The second observation is that, for limit suitable sequences B, the model $\mathbb{C}_B$ is definable in the model $M_B[G]$ constructed in Lemma 4.30.

Lemma 4.39. *Suppose that B is a limit suitable sequence and G is the set constructed above, and let B° be the set of heads of gaps of B. Then $\mathbb{C}_B$ is a submodel of $M_B[G]$, definable in $M_B[G]$ using the parameters G, B and B°.*

Proof. By Definition 3.10, $\mathbb{C}_B$ is equal to the set constructed by recursion over the set of ordinals of $M_B|\Omega$, using as parameters for the successor step the set $D = \bigcup\{\, [\Omega \cap M_{\tilde{B}}]^\omega \mid \tilde{B} \subseteq B \ \ \& \ \ \tilde{B} \text{ is suitable}\,\}$. Thus we need to verify that D is definable in the indicated parameters. Now any member of D may be written as in the form $\langle\, i_\Omega(f_n)(\vec{\beta}_n) \mid n \in \omega \,\rangle$ where $\vec{\beta}_n$ is a finite sequence of generators in $\tilde{B}$ for some suitable $\tilde{B} \subseteq B$. If we write $\vec{B} = \langle\, \beta_m \mid m \in \omega \,\rangle$ for $\bigcup_{n\in\omega} \vec{\beta}_n$, then each β_m is a generator for some $\bar{\kappa}_{\gamma_m}$ so that $\{\, \bar{\kappa}_{\gamma_m} \mid m \in \omega \,\} \subseteq \tilde{B}$. There will be a sequence $\vec{\xi} \in M_B$ so that in $M_B[G]$, $\gamma_m = h_{\zeta,\gamma_m}(\xi_m)$; furthermore $\vec{\gamma}$ satisfies, in $M_B[G]$, the condition

$$\forall \lambda \in B^\circ \sup(\vec{\gamma} \cap \lambda) < \lambda. \tag{10}$$

Thus it will be sufficient to show that for any sequences $\vec{\xi} \in [\Omega, \Omega^+)]^\omega \cap M_B$ and $\vec{\gamma} \in [B]^\omega$ satisfying the condition (10), the sequence $\langle\, h_{\zeta,\gamma_m}(\xi_m) \mid m \in \omega \,\rangle$ is in D.

Now since $\vec{\xi} \in M_B$, there is a function $f \in M$ and a finite sequence $\vec{\mu}$ of generators such that $\vec{\xi} = i(f)(\vec{\mu})$. Since $\vec{\mu}$ is finite, we can assume, by expanding $\tilde{B}$ if necessary, that $\vec{\mu}$ and hence $\vec{\xi}$ is in $M_{\tilde{B}}$. Now let $[s] \in G$ be a condition with $\xi_m \in \operatorname{domain}(a^{s,\zeta}_{\zeta,\gamma_m}) \cup \operatorname{domain}(f^{s,\zeta}_{\zeta,\gamma_m})$ for all $m \in \omega$. This partitions ω into three subsets:

$$A_0 = \{\, m \in \omega \mid \xi_m \in \operatorname{domain}(a^{s,\zeta}_{\zeta,\gamma_m}) \,\}$$
$$A_1 = \{\, m \in \omega \mid \xi_m \in \operatorname{domain}(f^{s,\zeta}_{\zeta,\gamma_m}) \ \ \& \ \ f^{s,\zeta}_{\zeta,\gamma_m}(\xi_m) \in \Omega \,\}$$
$$A_2 = \{\, m \in \omega \mid \xi_m \in \operatorname{domain}(f^{s,\zeta}_{\zeta,\gamma_m}) \ \ \& \ \ f^{s,\zeta}_{\zeta,\gamma_m}(\xi_m) \text{ has the form } h_{\gamma'_m,\gamma_m}(\xi'_m). \,\}$$

The set $\langle\, h_{\zeta,\gamma_m}(\xi_m) \mid m \in A_0 \,\rangle$ is clearly in D, and $\langle\, h_{\zeta,\gamma_m}(\xi_m) \mid m \in A_1 \,\rangle$ is a member of $M_{\tilde{B}}$ and hence is in D. Now for $m \in A_2$ we have $f^{s,\zeta}_{\zeta,\gamma_m} = h_{\gamma'_m,\gamma_m}(\xi'_m)$ as a formal expression. For those m such that $\xi'_m \notin [\bar{\kappa}_{\gamma'_m}, \bar{\kappa}^+_{\gamma'_m})$ we have $h_{\zeta,\gamma_m}(\xi_m) = 0$ by definition, and since $\langle\, \xi_m \mid m \in M_B \,\rangle$, the genericity of G ensures that $\{\, m \in A_2 \mid \xi'_m \in \left[\bar{\kappa}_{\gamma'_m}, \bar{\kappa}^+_{\gamma'_m}\right) \,\}$ is finite. □

4.7 Proof of the Main Lemma

The purpose of this subsection is to prove Lemma 3.11 with the simplifying assumption that $\kappa = \kappa_0$ is a member of the limit suitable set B. The following Subsection 4.8 will show how to remove this assumption, in the process giving the technique for proving the stronger result Theorem 3.6.

Before beginning the proof, we state two general facts about iterated ultrapowers. Both are well known facts, but we need to verify that they are valid in the context in which they will be used.

Lemma 4.40. *Suppose $\kappa' \leqslant \kappa$, E' is an extender on κ', and E is an extender on κ such that $[\eta]^\kappa \subseteq \mathrm{Ult}(V, E)$ for all $\eta < \mathrm{length}(E)$. Suppose further that if $\kappa' = \kappa$ then $E' \lhd E$, and if $\kappa' < \kappa$ then $\mathrm{length}(E') < \kappa$. Then the ultrapowers by E and E' commute, that is, $i^{i^{E'}(E)} \circ i^{E'} = i^{E'} \circ i^{E}$.*

Proof. This is a standard result in the case that E and E' are both ultrafilters: if $\kappa' < \kappa$ then each of the iterated ultrapowers is given by a single ultrapower by $E' \times E$ generated by the sets $X \subset \kappa$ which contain a rectangle $A \times B$ with $A \in E'$ and $B \in E$. Here, for $i^{E'} \circ i^{E}$, if $X_\alpha = \{\, \beta \in \kappa \mid (\alpha, \beta) \in X \,\}$ then $A = \{\, \alpha \in \kappa' \mid B_\alpha \in E \,\}$ and $B = \bigcap_{\alpha \in A} B_\alpha$, while for $i^{i^{E'}(E)} \circ i^{E'}$, A is such that $B = \{\, \alpha \mid \kappa \mid A = \{\, \beta \mid (\alpha, \beta) \,\} \in X \,\} \in E$. For the case $\kappa' = \kappa$, the rectangle $A \times B$ is replaced with a triangle $\{\, (\alpha, \beta) \mid \alpha < \kappa \wedge \beta < g_\kappa(\beta) \,\}$ where g_κ is a function such that $\kappa = [g_\kappa]_E$.

Now for extenders E and E', we write E_a for the ultrafilter $\{\, x \subseteq \kappa \mid a \in i^E(x) \,\}$. Then

$$\mathrm{Ult}(\mathrm{Ult}(V, E), E') = \operatorname*{dir\,lim}_{\substack{a \in \mathrm{supp}(E) \\ a' \in \mathrm{supp}(E')}} \mathrm{Ult}(\mathrm{Ult}(V, E_a), E_{a'})$$

and hence can be mapped into $\mathrm{Ult}(\mathrm{Ult}(V, E'), i^{E'}(E))$ by

$$\begin{aligned} \mathrm{Ult}(\mathrm{Ult}(V, E_a), E'_{a'}) &\cong \mathrm{Ult}\left(\mathrm{Ult}(V, E'_{a'}), i^{E'_{a'}}(E)_{i^{E'_{a'}}(a)}\right) && (11) \\ &\prec \operatorname*{dir\,lim}_{\substack{a' \in \mathrm{supp}(E') \\ a \in \mathrm{supp}(i^{E'}(E))}} \mathrm{Ult}(\mathrm{Ult}(V, E'_{a'}, i^{E'}(E)_a)) && (12) \\ &= \mathrm{Ult}(\mathrm{Ult}(V, E'), i^{E'}(E)). \end{aligned}$$

It remains to verify that the map is onto. This is immediate, except that in line (11) the only subscripts included for $i^{E'_{a'}}(E)$ are elements in the range of $i^{E'}$, where as in line (12) any $a \in \operatorname{supp}(i^{E'}(E))$ is allowed. Now fix any such a, and let g_a be such that $a = [g_a]_{E',a;}$ (enlarging a', if necessary). Then $\operatorname{Ult}(\operatorname{Ult}(V, E'_{a'}, i^{E'}(E)_a)) \subseteq \operatorname{Ult}\operatorname{Ult}(V, E'_{a'}), i^{E'}(E)_{i^{E'}(g_a)}$, which is an instance of the right side of line (11). □

The extender used in this proof do not quite satisfy the hypothesis of Lemma 4.40 as stated, as they are not closed in ω_1 sequences cofinal in $\operatorname{supp}(E)$. However all the ultrapowers involved are ω_2-complete, so the iteration maps are all continuous there, and so the conclusion applies. In the sequel, whenever we refer to an iterated ultrapower we will mean one by extenders to which Lemma 4.40 applies.

Corollary 4.41. *Any iterated ultrapower of a model M of set theory is equal to an iterated ultrapower obtained by reordering the extenders used so that the critical points are strictly increasing.*

Proof. Since the iterated ultrapowers are the direct limits of the finite subiterations, it is enough to show this for finite iterated ultrapowers, but this is a simple induction from Lemma 4.40. □

Corollary 4.42. *Suppose that $M \xrightarrow{k_0} M_0$ and $M \xrightarrow{k_1} M_1$ are iterated ultrapowers, with every extenders used in k_0 having critical point less than that of k_1, and with* $\operatorname{length}(k_1) < \operatorname{crit}(k_0)$*. Then $k_0(k_1)(\nu) \geqslant k_1(\nu)$ for every ordinal ν.*

Proof. Set κ equal to the smallest critical point of an extender in k_1 and set δ equal to the supremum of these critical points. Then we can regard k_1 as given by a single extender on the power set of δ, with $\operatorname{supp}(k_1) = \bigcup_{E \text{ in } k_1} \operatorname{supp}(E)$ and having all constituent ultrafilters κ-complete.

Define the extender E by $E_a = \{\, x \subseteq \delta \mid a \in k_1(x) \,\}$ for $a \in \operatorname{supp}(k_1)$. We will show that for every $g\colon \delta \to \Omega$ in M and $a \in \operatorname{supp}(k_1)$ there is $\bar{g}$ in M_0 such that $\{\, \nu \mid g(\nu) = \bar{g}(k_1(\nu)) \,\} \in E_a$. This will give an order preserving mapping from the ordinals of $\operatorname{Ult}(M, k_1)$ into those of $\operatorname{Ult}(M_0, k_0(k_1))$, proving the Corollary.

Given g, for each $\nu < \delta$ choose (g_ν, b_ν) with $b_\nu \in \operatorname{supp}(k_0)$ so that $\nu = k_0(g_\nu)(b_\nu)$. Define $\bar{g}_\beta(\nu) = g_\nu(\beta)$. By the κ-completeness of E_a, there is b so that $B = \{\, \beta \in \delta \mid b_\beta = b \,\} \in E_a$. Set $\bar{g} = k_0(\langle\, \bar{g}_\beta \mid \beta < \lambda \,\rangle)(b)$. Then for all $\nu \in B$, we have $\bar{g}(k_0(\nu)) = k_0(\langle\, \bar{g}_\beta \mid \beta < \lambda \,\rangle)(b)(k_0(\nu)) = k_0(\langle\, g_\beta(\nu) \mid \beta < \lambda \,\rangle(b) = g(\nu)$. □

The next Corollary is a slight generalization of a classic result of Kunen [Kun71]:

Corollary 4.43. *For each ordinal α there is a finite set d_α of regular cardinals such that $k(\alpha) = \alpha$ for any iterated ultrapower k such that* (i) *$k(d_\alpha) \restriction d_\alpha$ is the identity,* (ii) *the set of critical points of extend ers in k is bounded in α, and* (iii) *if k is factored $k = k_1 \circ k_0$, where all extenders in k_1 have critical points greater than any critical point of any extender in k_0, then there is k_1' so that $k_1 = k_0(k_1')$.*

Furthermore, this remains true for k in a generic extention, provided that if $k\colon V \xrightarrow{k_0} N_0 \xrightarrow{k_1} N_1$ is any factorization of k then $k_0(k_1)$ is in a generic extension of N_1.

Proof. The sets d_α are defined by recursion on α: we assume that the set $d_{\alpha'}$ has been defined and satisfy the conclusion for all $\alpha' < \alpha$ and will use this to define d_α.

Say that a pair (d, δ) *fixes* α if (i) d is a finite set of cardinals, (ii) $\delta \in \alpha$, (iii) the conclusion of Corollary 4.43 is true with d_α replaced with $d \cup \delta$. Clearly the pair $(\{\mathrm{cf}(\alpha), \alpha\}$ fixes α; we want to show that there is d so that $(d, 0)$ fixes α. Let δ be least such that there is d such that (d, δ) fixes α. We will assume that $\delta > 0$ and reach a contradiction.

Claim 4.43.1. *Suppose that $\delta' < \delta$ and the iteration k is a witness that (d, δ') does not fix α. Then the initial segment k_0 of k with critical points below δ also is such a witness.*

Proof of Claim. Write $k = k_1 \circ k_0$, with $k\colon V \xrightarrow{k_0} N_0 \xrightarrow{k_1} N_1$. The requirement in the hypothesis that $k_1 = k_0(k_1')$ implies that $k_1 \in N_0$, and by elementarity N_0 satisfies that $(k_0(d), k_0(\delta))$ fixes $k_0(\alpha)$, but $k_1(k_0(d)) = k(d) = d$, and the critical points of k_1 are all above δ, so $k_1(k_0(\alpha)) = k_0(\alpha)$. Since $k(\alpha) > \alpha$, it follows that $k_0(\alpha) > \alpha$. □

Claim 4.43.2. *There is d such that for any witness k that (d, δ') does not fix α, there is an initial segment k_0 of k with critical points bounded in δ which also is such a witness.*

Proof of Claim. Since the Claim is immediate if $\delta = \alpha$, we can assume that d_δ is defined. We can also assume that α is not a regular cardinal, as in that case we could take $d_\alpha = \{\alpha\}$. Let $d \supseteq \{\mathrm{cf}(\alpha), \mathrm{cf}(\delta)\} \cup d_{\mathrm{cf}(\alpha)} \cup d_\delta$ so that $d_\nu \subseteq d$ for all $\nu \in d$. Since $\mathrm{cf}(\alpha) \in d$, the embedding k is continuous at α; thus there is some $\alpha' < \alpha$ such that $k(\alpha') \geqslant \alpha$. Set $\xi = \max(\delta \cap d_{\alpha'} \cup \bigcup_{\nu \in d_{\alpha'} \cup d_\delta} d_\nu)$. Then $k(\xi) < \delta$, since the choice of d implies that $k(\delta) = \delta$. Let k_0 be the initial segment of k with critical points below $k(\xi)$ and factor $k = k_1 \circ k_0$. As in the proof of Claim 4.43.1, $k_1(k_0(\alpha')) = k_0(\alpha')$ and hence $k_0(\alpha') = k(\alpha') > \alpha$. □

Now we are ready to conclude the proof. Fix $k_0\colon V \to M_0$. Set $\xi = \sup(d_\alpha^{N_0})$, which is defined because $\alpha < k_0(\alpha)$. I claim that $\delta' = \xi$ works. Otherwise let $k_1\colon V \to M_1$ have all critical points in the interval $[\xi, \xi')$ for some $\xi' < \delta$ be such that $k_1(\alpha) > \alpha$. Then by Corollary 4.42 $k_0(k_1)(\alpha) \geqslant k_1(\alpha) > \alpha$, contradicting the fact that $k_0(k_1) \restriction d_\alpha^{M_0}$ is the identity.

To see why the "furthermore" clause of the Corollary is true, note that a critical point was that "there is a witness that (d, δ) does not fix α" is a first order statement. □

We are now ready to continue with the proof of Lemma 3.11. As was stated earlier, the argument uses an induction on the lexicographic ordering of pairs (ι, φ) to prove that for all limit suitable sequences B and all x in $\mathbb{C}_\iota \cap \mathbb{C}_B$,

$$\mathbb{C}_B \models_{\mathbb{C}_\iota} \varphi(x) \quad \text{if and only if} \models_{\mathbb{C}_\iota} \varphi(x). \tag{13}$$

Here and for the remainder of the paper, if P is a model of set theory then we write $P \models_{\mathbb{C}_\iota} \sigma$ to mean that $(\mathbb{C}_\iota)^P \models \sigma$.

The statement (13) implicitly uses the induction hypothesis by assuming that $\mathbb{C}_B \subseteq \mathbb{C}$. This is not literally true; however the induction hypothesis implies that $\mathbb{C}_B$ is isomorphic to a submodel of $\mathbb{C}_\iota$ by the map defined recursively by taking a set $\{\, y \in \mathbb{C}_{\iota'} \mid \mathbb{C}_{\iota'} \models \varphi(y, a) \,\}^{\mathbb{C}_B} \in (\mathbb{C}_\iota)^{\mathbb{C}_B}$ to the set having the same definition in $\mathbb{C}$. For the rest of this section we will identify these two sets.

We will need an additional induction hypothesis in order to carry out the proof. Because it is rather technical and uses notation which will be developed during the proof of the induction step for Lemma 3.11, we defer its statement, as Lemma 4.49, until it is needed to complete that proof.

By standard arguments, the only problematic part of the proof of the induction step for Lemma 3.11 is the assertion that the existential quantifier is preserved downwards: We assume that $\psi(x, y)$ is a formula which satisfies (13), and want to prove that

$$\forall x \in \mathbb{C}_B \left(\models_{\mathbb{C}_\iota} \exists y\, \psi(x, y) \implies \mathbb{C}_B \models_{\mathbb{C}_\iota} \exists y\, \psi(x, y)\right). \tag{14}$$

Since the basic problem in the proof is dealing with gaps in B, it will be useful to introduce some notation for them. A *gap* of B is a maximal nonempty interval of $I \backslash B$. Each gap in a limit suitable set B is a half open interval $[\sigma, \delta) \cap I$, where σ is the supremum of an ω-sequence of members of B, and δ is either $\min(B \backslash \sigma)$ or Ω. In either case we will call δ the *head of the gap* and the final ω-sequence of $B \cap \delta$, excluding the limit point, the *tail.* We will also refer to any terminal subsequence of it as a tail. Note that if $\delta' = \sup((\{0\} \cup I) \cap \sigma \backslash B)$ then the half open interval $I \cap [\delta', \sigma)$ is contained in B. We will call this a *block* of B.

Recall that if B is limit suitable then $\mathbb{C}_B$ is is defined to be the union over all suitable subsequences $\tilde{B} \subset B$ of $\mathbb{C}_{\tilde{B}}$. We will concentrate on suitable subsequences $\tilde{B}$ which are maximal in the sense that (i) every head δ of a gap of B is in $\tilde{B}$ (and therefore is also the head of a gap of $\tilde{B}$), (ii) these are the only gaps of $\tilde{B}$, and (iii) if δ is the head of a gap of B then $\max(\tilde{B} \cap \delta)$ is a member of the tail of that gap in B. Call a set $b \subseteq B$ a *tail traverse* of B if it contains exactly one point of the tail of each gap of B. If we write D for the set of heads of gaps of B, then every tail traversal b corresponds to a maximal suitable subset $\tilde{B} = B \backslash \bigcup\{\, [\lambda, \delta) \mid \delta \in D \;\;\&\;\; \lambda = \max(B \cap \delta) \,\} = \bigcup\{\, [\delta', \lambda) \mid \delta' \in \{0\} \cup D \backslash \{\Omega\} \;\;\&\;\; \lambda = \min(b \backslash \delta') \,\}$; conversely, if $\tilde{B}$ is a maximal suitable subset then $b = \{\, \min(\delta \cap B \backslash \tilde{B}) \mid \delta \in D \,\}$ is a tail traversal. This divides each block $B \subseteq [\delta', \delta)$ of B into three parts: the initial segment $[\delta', \delta) \cap \tilde{B} = [\delta', \lambda) \cap I$, the singleton $\{\lambda\} = b \cap [\delta', \delta)$, and the tail $B \cap (\lambda, \delta)$ of this block above λ.

Now suppose that $\varphi(x)$ is the formula $\exists y\,\psi(x,y)$ and is true in $\mathbb{C}_\iota$, and that B is a limit suitable sequence with $x \in \mathbb{C}_B$. Fix a tail traversal b of B such that $\{x,\iota\} \subseteq \mathbb{C}_{\tilde{B}}$, where $\tilde{B}$ is the suitable subsequence of B determined by b. Pick y so $\models_{\mathbb{C}_\iota} \psi(x,y)$ and let $B' \supseteq B$ be a limit suitable sequence with $y \in \mathbb{C}_{B'}$. By the induction hypothesis $\mathbb{C}_{B'} \models_{\mathbb{C}_\iota} \psi(x,y)$.

We will define an iteration map k and an isomorphism σ as in Diagram (15).

$$
\begin{tikzcd}
& & M_{B'} & & M_{B'}\upharpoonright\vec{\eta} \arrow[ll, hook'] \arrow[d, squiggly, "\sigma"] \\
M \arrow[r, "i_\Omega"] & M_{\tilde{B}} \arrow[r, hook] \arrow[urr, hook] & M_B \arrow[r, "k"] \arrow[u, hook] & M_k & M_k\upharpoonright\vec{\eta} \arrow[l, hook']
\end{tikzcd}
\tag{15}
$$

Here the wavy arrow, $\rightsquigarrow$, is used to indicate an isomorphism.

The map k will be an iterated ultrapower, definable in $M_B[c]$ from a countable sequence $c \in M_B$ of ordinals. The indiscernibles added by this iteration will be used for two distinct purposes: The first is to provide targets onto which σ can map members of $B' \backslash B$, and the second is to emulate the gaps of $B' \backslash B$ by adding blocks of indiscernibles of order type ω_1.

The map σ must also be defined on generators belonging to members of $B' \backslash B$. Since the extenders used in k will be members of M_B, they cannot provide enough generators to accomodate all of those in $M_{B'}$. Thus a submodel $M_{B'}\upharpoonright\vec{\eta}$ of $M_{B'}$ will be used which can be accommodated in M_k, but is large enough that $M_{\tilde{B}} \cup \{y\} \subseteq M_{B'}\upharpoonright\vec{\eta}$. Corollary 4.43 will be used to ensure that the restrictions of k and σ to ordinals in the suitable submodel $M_{\tilde{B}}$ are the identity.

Since the iteration k can be defined in $M_B[c]$, and thus in the generic extension of M_B described in Subsection 4.6, the models M_B and M_k have the same ordinals and the same associated Chang model $\mathbb{C}_B = \mathbb{C}_k$. Thus Diagram (15) induces the following diagram:

$$
\begin{tikzcd}
& \mathbb{C}_{B'} & & \mathbb{C}_{B'}\upharpoonright\vec{\eta} \arrow[ll, hook'] \arrow[d, squiggly, "\sigma"] \\
\mathbb{C}_{\tilde{B}} \arrow[r, hook] \arrow[ur, hook] & \mathbb{C}_B \arrow[r, "k"] \arrow[u, hook] & \mathbb{C}_k = \mathbb{C}_B & \mathbb{C}_k\upharpoonright\vec{\eta} \arrow[l, hook']
\end{tikzcd}
\tag{16}
$$

With this machinery in place, we will be able to quickly complete the proof: we are assuming $\models_{\mathbb{C}_\iota} \psi(x,y)$, so by the induction hypothesis $\mathbb{C}_{B'} \models_{\mathbb{C}_\iota} \psi(x,y)$. An easy proof will give Lemma 4.47, stating that $\mathbb{C}_{B'}\upharpoonright\vec{\eta} \prec \mathbb{C}_{B'}$, so $\mathbb{C}_{B'}\upharpoonright\vec{\eta} \models_{\mathbb{C}_\iota} \exists y\psi(x,y)$. Fix $y \in \mathbb{C}_{B'}\upharpoonright\vec{\eta}$ so that $\mathbb{C}_{B'}\upharpoonright\vec{\eta} \models_{\mathbb{C}_\iota} \psi(x,y)$. Since σ is an isomorphism, it follows that $\mathbb{C}_k\upharpoonright\vec{\eta} \models_{\mathbb{C}_\iota} \psi(x,\sigma(y))$.

The proof will be completed by showing that this implies $\mathbb{C}_k = \mathbb{C}_B \models \psi(x,\sigma(y))$, but this step is more difficult than the step using Lemma 4.47 for the upper level of the diagram. This argument uses the additional induction hypothesis alluded to earlier: Lemma 4.49 is a slightly generalized form of the needed fact which will conclude the proof of the induction step for Lemma 3.11. Finally, the full induction hypothesis, including the just proved fact that Lemma 3.11 holds for (ι, φ), will be used to prove that Lemma 4.49 also holds for (ι, φ); this will complete the proof of Lemmas 3.11 and 4.49, and thus (except for the assumption that $\kappa_0 \in B$) of Theorem 1.4.

We now give the details of the construction of Diagram (15). The four models on the left of the diagram have already been defined: B is the given limit suitable sequence, $\tilde{B} \subset B$ is a suitable subsequence with $x \in M_{\tilde{B}}$ which is characterized by a tail traversal b of B, and $B' \supseteq B$ is a limit suitable sequence with a witness y to $\exists y \psi(x, y)$. The following definition is more general than needed here. The added generality will be used in the proof of Lemma 4.49.

Definition 4.44. A *virtual gap construction sequence* for B is a triple $(b, \vec{\eta}, g)$ satisfying the following conditions: (i) b is a tail traversal of B, (ii) $\vec{\eta}$ is a sequence of countable ordinals with domain of the form $\{ (\lambda, \xi) \mid \lambda \in b \wedge \xi < \nu_\lambda \}$ for countable ordinals ν_λ, (iii) g is a set of pairs $(\lambda, \xi) \in \operatorname{domain}(\vec{\eta})$ with ξ a limit ordinal, and finally (iv) $\eta_{\lambda,\xi} > \operatorname{otp}(\{ z \in B \cup \operatorname{domain}(\eta) \mid z \lessdot (\lambda, \xi) \}$, where $(B \cup \operatorname{domain}(\eta))$ is the extension of the lexicographic order $\lessdot$ on $\operatorname{domain}(\vec{\eta})$ to $B \cup \operatorname{domain}(\vec{\eta})$ defined by setting $\lambda' \lessdot (\lambda, \xi) \lessdot \lambda$ whenever $(\lambda, \xi) \in \operatorname{domain}(\eta)$ and $\lambda' \in B \cap \lambda$.

Definition 4.45. We will say that $(b, \vec{\eta}, g)$ is a virtual gap construction sequence for B' over B if (i) B' and B are limit suitable sequences with $B' \supset B$, (ii) B' has the same order type as $(B \cup \operatorname{domain}(\eta), \lessdot)$, and, letting τ be the order isomorphism, there is a tail traversal b of B, and a tail traversal b' of those gaps in B' which are also in B, such that if $\tilde{B} \subseteq B$ is the associated suitable subsequence then (iii) $\tau \restriction \tilde{B}$ is the identity, and τ maps b' to b and the tail above each $\lambda' \in b'$ to the tail above $\tau(\lambda) \in b$, and furthermore, (iv) $g = \{ \tau(\gamma) \mid \gamma \text{ is the head of a gap in } B' \backslash B \}$.

The virtual gap construction sequence $(b, \vec{\eta}, g)$ for B' over B which will be used for the construction of Diagram (15) is represented in Figure 3 by the points in M_k, and the dotted lines connecting $M_{B'}$ and M_k in that figure correspond to the map τ. These are defined individually for each gap of B: Let $[\mu, \delta)$ be a gap of B and $[\delta', \mu)$ the corresponding block. In the case $B' \cap [\delta', \delta) = B \cap [\delta', \delta)$, then $\tau \restriction B' \cap [\delta', \delta)$ is the identity, $b' \cap [\delta', \mu)] = b \cap [\delta', \mu)$, and there are no members of $\operatorname{domain}(g)$ in the interval $[\delta', \delta)^{\lessdot}$.

Now assume that $\mu' = \sup(B' \cap \delta) > \mu$. Write λ for the member of b in $[\delta', \delta)$ and pick any member of the tail in B' of this gap as a member of b'. In accordance with clauses (iii) and (iv) of Definition 4.45, $\nu_\lambda = \operatorname{otp}([\lambda, \lambda') \cap B')$ and g is the set of $\tau(\gamma)$ such that γ is the head of a gap in $B' \cap (\lambda, \lambda')$. The function $\vec{\eta}$ is a constant function, with the constant value η chosen so that (i) $\eta \geqslant \omega^\omega \cdot \operatorname{otp}(B')$ and (ii) $y \in M_{B'} \restriction \vec{\eta}$, which is the submodel of $M_{B'}$ defined as follows:

Definition 4.46. If $(b, \vec{\eta}, g)$ is a virtual gap construction sequence for B' over B, then $M_{B'} \restriction \vec{\eta} = \{ j_\Omega(f)(a) \mid f \in M \wedge a \in [\mathcal{G}]^{<\omega} \}$ where $\mathcal{G}$ is the following set of generators: Let κ_ν be a member of B' and let $\beta = i_\nu(\bar\beta)$ be a generator belonging to κ_ν. Then

$$\beta \in \mathcal{G} \iff \big(\tau(\kappa_\nu) \in B \vee \big(\tau(\kappa_\nu) = (\lambda, \xi) \in \operatorname{domain}(\vec\eta) \wedge \bar\beta \in \operatorname{supp}(E_{\eta_{\lambda,\xi}})\big)\big).$$

Note that $M_{\bar B} \subseteq M_{B'} {\restriction} \vec\eta \prec M_{B'}$, and $M_{B'} = \bigcup_{\eta<\kappa^+} M_{B'} {\restriction} \vec\eta$, so $y \in M_{B'} {\restriction} \vec\eta$ for sufficiently large η.

Lemma 4.47. *If $(b, \eta_{\lambda,\xi}, g)$ is a virtual gap construction sequence for B' over B then $\mathbb{C}_{B'} {\restriction} \vec\eta \prec \mathbb{C}_{B'}$.*

Proof. A slight modification of the construction from Subsection 4.6 yields a $M_{B'} {\restriction} \vec\eta$-generic subset $G \subseteq i_\Omega(P(\vec E {\restriction} \operatorname{otp}(B'))$ so that $M_{B'} {\restriction} \vec\eta[G]$ is closed under countable sequences. The only change needed in the construction is the restriction of the range of the coordinate b_γ to $\operatorname{supp}(E_{\eta_{\lambda,\xi}})$ whenever $(\lambda, \xi) \in \operatorname{domain}(\vec\eta)$ and κ_γ is the ξth member of B' above λ.

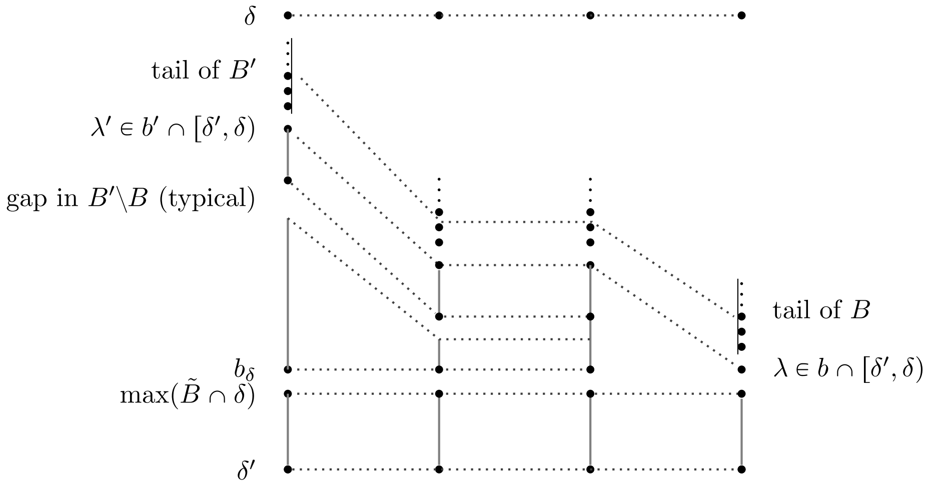
Now suppose φ is a formula, with parameters given by standard forcing names, which is true in $\mathbb{C}_{B'} {\restriction} \vec\eta$. By Lemma 4.38 there is a condition $([r], b)$ in the forcing R for $M_{B'} {\restriction} \vec\eta$ which establishes the parameters of φ such that $[r] \Vdash \varphi$. Now $([r], b)$ is also a condition in $R^{M_{B'}}$, so we can use Section 4.6 to yield a $M_{B'}$-generic subset G' of $i_\Omega(P(\vec E {\restriction} \operatorname{otp}(B'))$ which includes $i_\Omega([r])$ and establishes the same parameters. Hence φ holds in $\mathbb{C}^{M_{B'}[G']} = \mathbb{C}_{B'} {\restriction} \vec\eta$. □

Clause 4.44(iv) is used here to ensure that the enough of the image of E at each $\kappa_\nu \in B' \backslash B$ is present in $M_{B'} {\restriction} \vec\eta$ to construct the generic set as in section 4.6.

Now we want to complete the definition of the elements of Diagram (15) by defining k and σ. The restriction of σ to B' is determined by the map τ specified in the Definition 4.44 of a virtual gap construction sequence for B' over B: if $\tau(\gamma) \in B$ then $\sigma(\gamma) = k(\tau(\gamma))$, and if $\tau(\gamma) = (\lambda, \xi) \in \operatorname{domain}(\vec\eta)$ then $\sigma(\gamma)$ is the critical point of an ultrapower of λ in the iteration k. The restriction of σ to B' determines its restriction to the generators of $M_{B'} {\restriction} \vec\eta$, which determines in turn the remainder of σ.

Thus it will be sufficient to define the iteration k, which consists of an ultrapower by the image of E_{η_z} for each $z \in \operatorname{domain} \vec\eta$ and, in addition, for each member of g an iteration of length ω_1. For the latter we need to begin by choosing a sequence $\vec F$ of extenders in M: a suitable choice is to let F_ν be the least $\kappa^{+\nu}$-strong extender in M for each $\nu < \omega_1$. The two essential conditions that the choice of $\vec F$ must satisfy are (i) $\vec F \in \operatorname{Ult}(M, E)$ and (ii) $\vec F$ is cofinal among the extenders below E in M. The first clause is needed so that for each $(\xi, \lambda) \in g$, if γ is such that $\lambda = \kappa_\gamma$ then $i_\gamma(\vec F) \in M_B$; thus k is definable over M_B from a countable parameter in $M_B[G]$. This fact will be used to identify the ordinals of M_k with those of M_B. The second clause is needed so that the ordinal $k_{(\lambda,\xi)} \circ i_\gamma(\kappa)$, which will become $\sigma(\tau^{-1}(\lambda))$, depends only on g, or more precisely, on $\operatorname{otp}(\{\, \xi' < \xi \mid (\xi', \lambda) \in g \,\})$; this fact ensures (using Lemma 4.43) that the restriction of k to ordinals in M_B is similarly independent of the choice of B'. This fact will be needed for the proof of Lemma 4.49.

Here is the precise definition of k, which is illustrated by Figure 3.



$$M_{B'} \upharpoonright \vec{\eta} \overset{\sigma}{\rightsquigarrow} M_k \upharpoonright \vec{\eta} \hookrightarrow M_k \xleftarrow{k} M_B$$

Figure 3: The maps σ and k inside the block between δ' and δ which is associated with the gap in B headed by δ. The dotted lines represent the maps σ and k; the vertical lines represent intervals of I contained in the indicated models.

Definition 4.48. The iteration k is the direct limit of the sequence of embeddings $k_z \colon M_B \to M^*_z$ for z in the well ordering $(B \cup \mathrm{domain}(\vec{\eta}), \lessdot)$. In the following, δ is the head of a gap in B and δ' is the supremum of the set of the heads of gaps below δ (or $\delta' = 0$ if there are none). We assume that $k_z \colon M_B \to M^*_z$ has been defined for all $z \lessdot \delta'$. Let λ be the unique member of $b \cap [\delta', \delta)$.

1. $M_0 = M_B$.

2. If z is a limit point in the ordering $\lessdot$ and $z \notin g$ then $k_z \colon M_B \to M^*_z$ is the direct limit of the embeddings $k_{z'} \colon M_B \to M^*_{z'}$ for $z' \lessdot z$.

3. If $z = \tau \in B$ is the successor in I of τ' then $M^*_\tau = M^*_{\tau'}$ and $k_\tau = k_{\tau'}$. Hence $k_\tau = k_{\delta'}$ and $M^*_\tau = M^*_{\delta'}$ for all $\tau \in \tilde{B} \cap [\delta', \delta)$, and $k_\tau = k_{\lambda+1} = k_\delta$ and $M^*_\tau = M^*_{\lambda+1} = M^*_\delta$ for all τ in the tail $B \cap (\lambda, \delta)$ of B above λ.

4. If $z = (\lambda, \xi + 1) \in \mathrm{domain}(\vec{\eta})$, or if $z = \lambda$ and (λ, ξ) is its immediate predecessor in $\lessdot$, then $M^*_z = \mathrm{Ult}(M^*_{(\lambda,\xi)}, E^*_{\eta_{\lambda,\xi}})$ where, letting γ' be such that $\delta' = \kappa_{\gamma'}$, we write E^*_α for $k_{(\lambda,\xi)} \circ i_{\gamma'}(E_\alpha)$.

5. If $z = (\lambda, \xi) \in g$, then let $\bar{k}_z \colon M_B \to \overline{M}_z$ be the direct limit of the maps $\langle k_{z'} \mid z' \lessdot z \rangle$. Then k_z is the iterated ultrapower $i^{\vec{F}^*} \circ \bar{k}_z \colon M_B \to \overline{M}_z \to M^*_z$, where $\vec{F}^* = \bar{k} \circ i_{\gamma'}(\vec{F})$

This completes the definition of the map k. As pointed out earlier, this induces the definition of σ by setting $\sigma(\gamma)$ equal to the critical point of the ultrapower $i^{E*}_{\tau(z)+1}\colon M^*_{\tau(z)} \to M^*_{\tau(z)+1}$, and thus completes the definition of the maps of diagram (15). The extension to the Chang model illustrated in Diagram (16) is straightforward. We have already observed that the Chang model $\mathbb{C}_k$ built on M_k is the same as $\mathbb{C}_B$, giving the identity on the bottom of the diagram. Lemma 4.47 asserts that $C_{B'}\restriction\vec{\eta}$ is an elementary substructure of $\mathbb{C}_{B'}$, and $\sigma\colon C_{B'}\restriction\vec{\eta} \to \mathbb{C}_k\restriction\vec{\eta}$ is an isomorphism. It follows that $\mathbb{C}_k\restriction\vec{\eta} \models_{\mathbb{C}_\iota} \psi(x,\sigma(y))$, and we will be finished if we can conclude from this that that $\mathbb{C}_B \models_{\mathbb{C}_\iota} \psi(x,\sigma(y))$, and this is asserted by Lemma 4.49:

Lemma 4.49. *Suppose that $B \subseteq B'$ are limit suitable sequences and $\vec{\eta}$ is a virtual gap construction sequence for B' over B such that $\eta_{\lambda,\xi} \geqslant \omega^n \cdot \operatorname{otp}(B \cup \operatorname{domain}(\vec{\eta}), \lessdot)$ for all $(\lambda,\xi) \in \operatorname{domain}(\vec{\eta})$ and all $n \in \omega$. Let $k\colon M_B \to M_k$ be the virtual gap construction iteration, and let $\mathbb{C}_k\restriction\vec{\eta} \subseteq \mathbb{C}_k$ be as given in Diagram* (16)*. Then $\mathbb{C}_k\restriction\vec{\eta} \prec \mathbb{C}$.*

This is the promised addition to the induction hypothesis for Lemma 3.11, and concludes the proof of the induction step for that Lemma. It remains only to prove the induction step for Lemma 4.49:

Proof. As was stated earlier, this proof is a simultaneous induction along with Lemma 3.11: we assume the following two conditions on a pair (ι,φ) as an induction hypothesis:

1. Under the hypothesis of Lemma 3.11, $\mathbb{C}_B \models_{\mathbb{C}_{\iota'}} \theta(\vec{a}) \iff \models_{\mathbb{C}_{\iota'}} \theta(\vec{a})$ for any $\vec{a} \in C_{\iota'}$, provided that $\iota' < \iota$ or $\iota' = \iota$ and θ is a subformula of or equal to φ.
2. Under the hypothesis of Lemma 4.49, $\mathbb{C}_k\restriction\vec{\eta} \models_{\mathbb{C}_{\iota'}} \theta(\vec{a}) \iff \models_{\mathbb{C}_{\iota'}} \theta(\vec{a})$ for all $\vec{a} \in \mathbb{C}_k\restriction\vec{\eta}$, provided that $\iota' < \iota$ or $\iota' = \iota$ and θ is a proper subformula of φ.

The induction hypothesis used for Lemma 3.11 was the same, except that in the first clause the formula θ was required to be a proper subformula of φ. As in the proof of Lemma 3.11, the only problematic case with that in which $\varphi(x)$ is the formula $\exists y\, \psi(x,y)$.

Let B and $\vec{\eta}$ be as in Lemma 4.49, and let x be an arbitrary member of $C_k\restriction\vec{\eta}$ such that $\models_{\mathbb{C}_\iota} \exists y\psi(x,y)$. We need to show that $\mathbb{C}_k\restriction\vec{\eta} \models_{\mathbb{C}_\iota} \exists y\psi(x,y)$. By clause (1) of the induction hypothesis, $\mathbb{C}_B \models_{\mathbb{C}_\iota} \exists y\, \psi(x,y)$. Fix $y \in \mathbb{C}_B$ so that $\models_{\mathbb{C}_\iota} \psi(x,y)$. We now define an extension $\vec{\eta}'$ of the virtual gap construction sequence $\vec{\eta}$ such that $y \in C_B\restriction\vec{\eta}'$. The sequence $\vec{\eta}'$ will have the same sets b and g as $\vec{\eta}$, but the domain of $\vec{\eta}'$ will be enlarged by adding ω new elements as a new tail for each $(\lambda,\xi) \in g$. Thus, for each $\lambda \in b$ define a map t_λ with $\operatorname{domain}(t_\lambda) = \operatorname{length}(\vec{\eta}_\lambda)$ by

$$t_\lambda(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ t_\lambda(\xi') + 1 & \text{if } \xi = \xi' + 1, \\ \sup_{\xi' < \xi} t_\lambda(\xi') & \text{if } \xi \text{ is a limit and } (\lambda,\xi) \notin g \\ \sup_{\xi' < \xi} t_\lambda(\xi') + \omega & \text{if } (\lambda,\xi) \in g. \end{cases}$$

Now define $\vec{\eta}'$, using an ordinal $\eta' \in \omega_1$ to be determined shortly:

$$\operatorname{domain}(\vec{\eta}') = \{ (\lambda, \xi) \mid \xi < \sup \operatorname{range}(t_\lambda) \}$$
$$b^{\vec{\eta}'} = b^{\vec{\eta}}, \text{ and } g^{\vec{\eta}'} = \{ (\lambda, t_\lambda(\xi) \mid (\lambda, \xi) \in g^{\vec{\eta}} \}, \text{ and}$$
$$\eta'_{\lambda,\xi} = \begin{cases} \eta_{\lambda,\xi'} & \text{if } \xi = t(\xi') \\ \eta' & \text{if } (\lambda, \xi) \notin \operatorname{range}(t). \end{cases}$$

As in the choice of η, the two conditions on η' are that (i) $\eta' \geqslant \omega^n \cdot \operatorname{otp}(B \cup \operatorname{domain}(\vec{\eta}'), \lessdot)$ for each $n \in \omega$, and (ii) $y \in \mathbb{C}_k \restriction \vec{\eta}'$. Note that the first condition implies that $\vec{\eta}'$ satisfies the hypothesis of Lemma 4.49: if $n \in \omega$ and $\xi = t_\lambda(\xi')$ then

$$\begin{aligned} \eta'_{\lambda,\xi} = \eta_{\lambda,\xi'} &> \omega^{n+1} \cdot \operatorname{otp}(B \cup \operatorname{domain}(\vec{\eta}), \lessdot) \\ &= \omega^n \cdot \omega \cdot \operatorname{otp}(B \cup \operatorname{domain}(\vec{\eta}), \lessdot) \geqslant \omega^n \cdot \operatorname{otp}(B \cup \operatorname{domain}(\vec{\eta}'), \lessdot). \end{aligned}$$

The second condition will be satisfied by any sufficiently large η', since $\mathbb{C}_B = \mathbb{C}_k = \bigcup_{\eta' < \omega_1} \mathbb{C}_k \restriction \vec{\eta}'$.

For the remainder of the proof we refer to Diagram (17). The inner rectangle is the same as Diagram (15). The map τ is determined by using the map $(\lambda, \xi) \mapsto (\lambda, t_\lambda(\xi))$ to map the generators of indiscernibles from $\vec{\eta}$ into those of $\vec{\eta}'$. As with Diagrams (15) and (16), Diagram (17) induces a similar diagram for the corresponding Chang models.

$$\begin{tikzcd}
 & M_{B''} & & M_{B''} \restriction \vec{\eta}' \arrow[ll, hook'] \arrow[ddd, squiggly, "\sigma'"] \\
M_{B'} \arrow[ur, hook] & & M_{B'} \restriction \vec{\eta} \arrow[ll, hook'] \arrow[ur, hook] \arrow[d, squiggly, "\sigma"] & \\
M_B \arrow[u, hook] \arrow[r, "k"] \arrow[dr, "k'"'] & M_k \arrow[d, "\tau"] & M_k \restriction \vec{\eta} \arrow[l, hook'] \arrow[dr, hook, "\tau"] & \\
 & M_{k'} & & M_{k'} \restriction \vec{\eta}' \arrow[ll, hook']
\end{tikzcd} \tag{17}$$

We claim that $\tau \restriction (\mathbb{C}_k \restriction \vec{\eta})$ is the identity. First, Lemma 4.43 implies that the restriction of τ to the ordinals of $M_k \restriction \vec{\eta}$ is the identity. Now every member $\mathbb{C}_k \restriction \vec{\eta}$ is represented by a term $w = \{ z \in \mathbb{C}_{\iota'} \mid\models_{\mathbb{C}_\iota} \varphi(z, a) \}$, where $\iota' \in M_k \restriction \vec{\eta}$ and a is a sequence of ordinals from $M_k \restriction \vec{\eta}$. Thus $\tau(w)$ is represented by the same term in $\mathbb{C}_{k'} \restriction \vec{\eta}$. But $\mathbb{C}_k = \mathbb{C}_{k'} = \mathbb{C}_B$, so this term represents the same set w in $\mathbb{C}_{k'}$.

Now define B'' to be B' together with the next ω-many members of I from each of the gaps of B' which are not gaps of B. The right-hand trapezoid commutes, and in particular $(\sigma')^{-1}(x) = \sigma^{-1}(x)$. Now $\mathbb{C}_{k'} \restriction \vec{\eta}' \models_{\mathbb{C}_\iota} \exists y \psi(x, y)$, and since σ' is an isomorphism it follows that $\mathbb{C}_{B''} \restriction \vec{\eta}' \models_{\mathbb{C}_\iota} \exists y \psi(\sigma^{-1}(x), y)$. It follows by Lemma 4.47 that $\mathbb{C}_{B''}$ satisfies the same formula, and by the induction hypothesis Lemma 3.11 for (ι, φ) it follows that $\underset{\mathbb{C}_\iota}{\models} \exists y \psi(\sigma^{-1}(x), y)$. By another application of the same induction hypothesis it follows that $\mathbb{C}_{B'}$ satisfies the same formula, and by Lemma 4.47 again $\mathbb{C}_{B'} \restriction \vec{\eta}$ does as well. Finally, it follows that $\mathbb{C}_k \restriction \vec{\eta} \underset{\mathbb{C}_\iota}{\models} \exists y \psi(x, y)$, as required. $\square$

This completes the proof of Lemma 3.11, and hence of Theorem 1.4, except for the case that $\kappa_0 \notin B$. This is dealt with in the next section.

4.8 Dealing with finite exceptions and $\kappa_0 \notin B$

In the last subsection we assumed that $\kappa_0 = \kappa$ is a member of B; here we indicate how this extra assumption can be eliminated. The same argument supports the possibility of finitely many exceptions in the statement of Theorem 3.6.

The problem is that since $\kappa_0 \notin B$, the smallest member of B' may be smaller than the smallest member of B. This invalidates the definition of the map k_η in Diagram (15). Now suppose that $B = \{\, \lambda_\nu \mid \nu \leqslant \zeta \,\}$ is a limit suitable set with $\lambda_0 > \kappa_0$, that $x \in \mathbb{C}_B$, and that $\mathbb{C} \models \varphi(x)$. We want to show that $\mathbb{C}_B \models \varphi(x)$. Since B is limit suitable, $\lambda_0 > \kappa_\omega$. Let $B' = B \cup \{\, \kappa_n \mid n < \omega \,\}$. Since B' is also limit suitable and $\kappa_0 \in B'$, the version of Theorem 1.4(2) already proved implies that $\mathbb{C}_{B'} \models \varphi(x)$.

Now let $G \subset i_\Omega(P(\vec{E}\restriction\delta)/\leftrightarrow)$ be the $M_{B'}$-generic set constructed in Section 4.6 and consider the set $G \cap M_B$. By appealing the the factorization of $P(\vec{E}\restriction\delta)/\leftrightarrow$ given by Proposition 4.14, we can regard $G \cap M$ as a subset of $G' \times G'' \subset i_{\lambda_0}(P(\vec{E}\restriction\omega)/\leftrightarrow) \times R$. The set G'' is a M_B-generic subset of R, which is essentially $i_\Omega(P(\vec{E})\restriction(\omega, \zeta)/\leftrightarrow)$ with some additional Cohen subsets, and G' is an M_B-generic subset of the direct forcing order, $(i_{\lambda_0}(P(\vec{E}\restriction\omega)/\leftrightarrow, \leqslant^* /\leftrightarrow)$. Since x is in M_B, it has a name in the forcing over $M_{B'}$ which not involve any of the indiscernibles $\bar{\kappa}_n$ for $n \in \omega$, and because $\exists y \psi(x, y)$ is true in $\mathbb{C}_{B'}$, there is a condition $[s] \in G'$ which forces over $M_B[G'']$ that $\exists y \psi(x, y)$. Thus, extending G' to a M_B-generic subset $\bar{G}$ of the forcing order $i_{\lambda_0}(P(\vec{E}\restriction\omega)/\leftrightarrow), \leqslant)$ will give a model $\mathbb{C}_\iota^{M_B[G''][\bar{G}]}$ which satisfies $\exists y \psi(x, y)$. It is also true that any such generic subset will yield an unbounded set of indiscernibles in $\lambda_0 \cap M_B = \kappa_0$, and hence will collapse ω_1. However, $\mathbb{C}_B^{M_B[G''][\bar{G}]}$ is defined in $M_B[G''][\bar{G}]$ by treating its set $\{\, \bar{\kappa}_n \mid n \in \omega \,\} \cup B$ of indiscernibles as a limit suitable set with $\lambda_0 = \bar{\kappa}$ as the head of a gap. This means that the members of $\mathbb{C}_B^{M_B[G''][\bar{G}]}$ are the denotations of standard names, using as parameters sequences of ordinals which are bounded in κ_0. It follows that $\mathbb{C}_B^{M_B[\bar{G}][G'']}$ is the same as $\mathbb{C}_B^{M_B[G]''}$, which is $\mathbb{C}_B$. Hence $\mathbb{C}_B \models \exists y \psi(x, y)$.

This concludes the proof of Lemma 3.11 and hence of Theorem 1.4, and the same argument can be used to prove the generalization Theorem 3.6. Note that it is critical to the argument that there are only a finite number of gaps (in this case, only one gap) in B which need to be dealt with, for otherwise $B' \backslash B$ would include infinitely many extra ω sequences, and $\mathbb{C}_{B'}$ would include

Prikry sequences obtained by taking finitely many from indiscernibles each such sequence, which are not in the extension $M_{B'}[G \cap M_B]$.

5 Questions and Problems

This study leaves a number of questions open. Two which were mentioned in the introduction essentially involve filling gaps in this paper:

Question 5.1. What is the large cardinal strength of a sharp for $\mathbb{C}$?

This question can be taken in either a coarse or fine sense. In the coarse sense, we make the following conjecture:

Conjecture 5.2. If there is an extender model $N = L(\mathcal{R})[\vec{E}]$ over the reals with an extender E on $\vec{E}$ of length $\kappa^{+(\omega+1)}$ in N, then there is a mouse for the Chang model.

If the conjecture is false, then it would be surprising if the coarse answer to Question 5.1 were not given by Theorem 1.4, that is, that the sharp for the Chang model is a $\mathcal{R}$-mouse $M = J_\rho(\mathcal{R})[\vec{E}]$, projecting to $\mathcal{R}$, such that $\vec{E}$ has a final extender E of length either $\kappa^{+(\omega+1)}$ or $\kappa^{+\omega_1}$ in M, where κ is the critical point of E. If we assume that this is correct, then we can state the finer version of Question 5.1:

Question 5.3. What is the height ρ of this mouse?

If $\lambda = \text{length}(E)$ then ρ cannot be smaller than the index of E in the sequence $\vec{E}$, which in the indexing of [MS94] is $(\rho^+)^M$. It seems plausible that this is sufficient.

We repeat here a second point which was raised in the introduction:

Question 5.4. Does the mouse asked for in the previous problems give a real sharp? That is, is there can the choice of terms which eliminates the need for restricted formulas in the Definition 1.3 of a sharp?

A solution to this question may require proving Conjecture 3.4 from the introduction, which asserts that $K(\mathcal{R})^{\mathbb{C}}$ is an iterated ultrapower of the model $M_\Omega|\Omega$.

The structure of this iteration $j\colon M_\Omega|\Omega \to K(\mathcal{R})^{\mathbb{C}}$ poses some interesting questions:

Question 5.5. Is $j(\lambda) = \lambda$ for every $\lambda \in I$ of cofinality ω?

Note that this would follow from an affirmative answer to Conjecture 5.2 by the results of Gitik used to prove Theorem 1.4(1). Also, the same argument shows that every cardinal of cofinality ω is measurable in $K(\mathcal{R})^{\mathbb{C}}$. On the other hand, Gitik's results which were adapted for our proof of Theorem 1.4(2) suggest the following as a converse:

Question 5.6. Does every measurable cardinal of $K(\mathcal{R})^{\mathbb{C}}$ have cofinality ω in V?

It seems likely that a positive solution to Question 5.4 would imply a positive answer to Question 5.6 by extending an ultrafilter embedding from a measurable cardinals of uncountable cofinality to an embedding of $\mathbb{C}$ into itself.

The remaining questions involve the ω_1-Chang model, the least model of ZF containing all ω_1-sequences of ordinals.

Question 5.7. Is it consistent that there is a sharp for the ω_1-Chang model $\mathbb{C}(\omega_1)$? If so, what is its strength?

Little is known about this. For the lower bound, Gitik's technique for recovering extenders from threads given by iterations of length ω_1 can be used to show that it implies the existence of a $\mathcal{P}(\omega_1)$-mouse with an extender of length ω_2. To obtain longer extenders from this technique would require having a covering lemma giving covering sets of size ω_1 in $K(\mathcal{P}(\omega_1))^{\mathbb{C}(\omega_1)}$ for sets of size ω_1 in the ω_1-Chang model; however all of the mice in $\mathbb{C}(\omega_1)$ contain $\mathcal{P}(\omega_1)$ and hence are larger than ω_1.

References

[Cha71] C. C. Chang. Sets constructible using $L_{\kappa\kappa}$. In *Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)*, pages 1–8. Amer. Math. Soc., Providence, R.I., 1971. 1

[Git02] Moti Gitik. Blowing up power of a singular cardinal—wider gaps. *Ann. Pure Appl. Logic*, 116(1-3):1–38, 2002. 4

[GK12] Moti Gitik and Peter Koepke. Violating the singular cardinals hypothesis without large cardinals. *Israel J. Math.*, 191(2):901–922, 2012. 1.2

[GM96] Moti Gitik and William J. Mitchell. Indiscernible sequences for extenders, and the singular cardinal hypothesis. *Ann. Pure Appl. Logic*, 82(3):273–316, 1996. 2, 2

[Kun71] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *J. Symbolic Logic*, 36:407–413, 1971. 4.7

[Kun73] Kenneth Kunen. A model for the negation of the axiom of choice. In *Cambridge Summer School in Mathematical Logic (Cambridge, 1971)*, pages 489–494. Lecture Notes in Math. Vol. 337. Springer, Berlin, 1973. 1

[Mag78] Menachem Magidor. Changing cofinality of cardinals. *Fundamenta Mathematicae*, 99(1):61–71, 1978. 4, 10, 13, 15

[Mer07] Carmi Merimovich. Prikry on extenders, revisited. *Israel J. Math.*, 160:253–280, 2007. 4, 4.6

[MS94] William J. Mitchell and John R. Steel. *Fine Structure and Iteration Trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994. 130 pages. 5